

# Objective Bayesian Analysis in Acoustics

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*Many scientists analyze the experimental data using well-understood models or hypotheses as a way of thinking like a Bayesian.*

## Introduction

Experimental investigations are vital for scientists and engineers to better understand underlying theories in acoustics. Scientific experiments are often subject to uncertainties and randomness. Probability theory is the appropriate mathematical language for quantification of uncertainties and randomness, and involves rules of calculation for the manipulation of probability values. This article introduces probability theory in application to recent research in acoustics, taking the Bayesian view of probability. Bayesian probability theory, centered upon Bayes' theorem, includes all valid rules of statistics for relating and manipulating probabilities; interpreted as logic, it is a quantitative theory of inference (Jaynes, 2003).

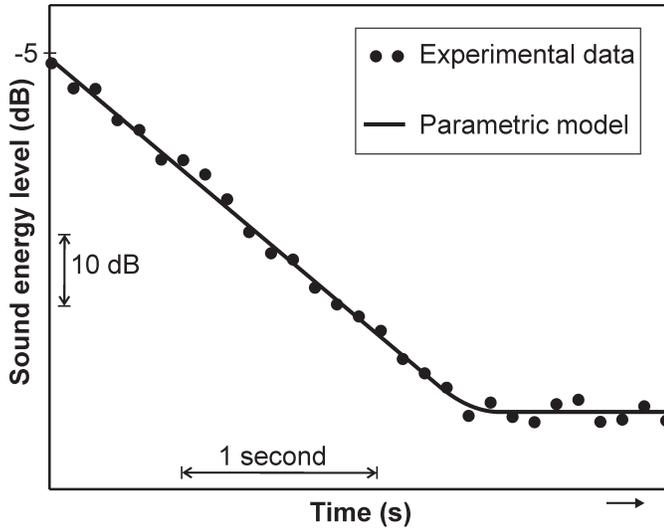
## Probability, Randomness

When considering the interpretation of probability, debate is now in its third century between the two main schools of statistical inference, the frequentist school and the Bayesian school. The frequentists consider probability to be a proportion in a large ensemble of repeatable observations. This interpretation has been dominant in statistics until recent decades. However, when expressing, for instance, the probability of Mars being lifeless (Jefferys and Berger 1992), the probability cannot be interpreted as frequencies of repeatable observations; there is only one Mars.

According to the Bayesian interpretation, probabilities represent quantitative measures of the state of knowledge or the degree of belief of an individual in the truth of a proposition (Caticha, 2012). In other words, the Bayesian view posits that a probability represents quantitatively how strongly we should believe that something is true. Probability is defined as a real valued quantity ranging between 0 and 1, and according to the Bayesian view, it represents the degree of appropriate belief, from total disbelief (0) to total certainty (1). Bayesian probability theory is not confined to applications involving large ensembles of repeatable events or random variables. Bayesian probability makes it possible to reason in a consistent and rational manner about single events. This interpretation is not in conflict with the conventional interpretation that probabilities are associated with randomness. Conceivably, an unknown influence may affect the process under investigation in unpredictable ways. When considering such an influence as "random," the randomness is expressed through the value of a quantity being unpredictable, uncertain, or unknown. With the Bayesian interpretation, probability can be used in wider applications, including but not restricted to situations in which the frequentist interpretation is applicable (Caticha, 2012). The methods of probability theory constitute consistent reasoning in scenarios where insufficient information for certainty is available. Thus, probability is the tool for dealing with uncertainty, lack of scientific data, and ignorance.

## Examples

An example in acoustics/seismic research is the investigation of earthquakes in California based on a limited number of globally deployed seismic sensors. Must we wait for the ensemble of repeatable, independent devastating earthquakes at the same location to occur in order to infer the location of their epicenter?



**Figure 1.** Room-acoustic energy decay process. Experimental data values collected in an enclosure are expressed as the sequence  $\mathbf{D} = [d_0, d_1, \dots, d_{K-1}]$  and are plotted on a logarithmic scale. The parametric model function is plausibly described by an exponential decay function  $\mathbf{H}(\Theta)$  having three parameters  $\Theta = [\theta_0, \theta_1, \theta_2]$ .

To introduce Bayesian analysis in acoustic studies, let us apply it to a data analysis task common not only in acoustic investigations but also in many scientific and engineering fields. This example begins with a room-acoustic measurement that records a dataset expressed as a sequence,  $\mathbf{D} = [d_0, d_1, \dots, d_{K-1}]$ . The data comprise a finite number of observation points as shown in **Figure 1**. These data represent a sound energy decay process in an enclosed space. Based on visual inspection of **Figure 1**, an experienced architectural acoustician will formulate the hypothesis that “the data probably represent an exponential decay.” This hypothesis can be translated into an analytical function of time ( $t$ ) as an “amplitude multiplied by (an exponential decay) + noise” or the so-called model

$$\mathbf{H}(\Theta) = \theta_0 + \theta_1 e^{-\theta_2 t} \quad (1)$$

This model contains a set of parameters  $\Theta = [\theta_0, \theta_1, \theta_2]$ , in which  $\theta_0$  is the noise floor,  $\theta_1$  is the initial amplitude and  $\theta_2$  the decay constant. The aim is to estimate this set of parameters  $\Theta$  so as to match the modeled curve (solid line) to the data points (black dots) as closely as possible. The data analysis task is to estimate the relevant parameters  $\Theta$  encapsulated in the model  $\mathbf{H}(\Theta)$ , particularly  $\theta_2$ , from the experimental observation  $\mathbf{D}$ . This is known as an inverse problem.

## Bayes' Theorem

Bayes' theorem was published posthumously in 1763 in *Philosophical Transactions of the Royal Society* (Bayes, 1763) through the effort of Richard Price, an amateur mathematician and a close friend of Reverend Thomas Bayes (1702-1761), two years after Bayes died. While sorting through Bayes' unpublished mathematical papers, Price recognized the importance of an essay by Bayes giving a solution to an inverse probability problem “in moving mathematically from observations of the natural world inversely back to its ultimate cause” (McGrayne, 2011). The general mathematical form of the theorem is due to Laplace (1812), who was also the first to apply Bayes' theorem to social sciences, astronomy, and earth science.

Nowadays, Bayes' theorem is straightforwardly derived from widely accepted axioms of probability theory, particularly through the product rule.

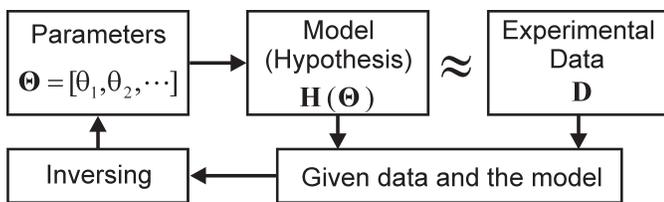
$$p(\Theta | \mathbf{D})p(\mathbf{D}) = p(\mathbf{D} | \Theta)p(\Theta). \quad (2)$$

In the **Equation 2**, the solidus (vertical bar) inside, e.g.,  $p(\Theta | \mathbf{D})$ , is a conditional symbol and it reads “probability of  $\Theta$  conditional on  $\mathbf{D}$ ” or, in short, “probability of  $\Theta$  given  $\mathbf{D}$ .” Bayes' theorem represents the principle of inverse probability (Jeffreys, 1965). The significance of Bayes' theorem, as recognized by R. Price, is that an initial belief expressed by  $p(\Theta)$  will be updated by objective new information,  $p(\mathbf{D} | \Theta)$ , about the probable cause of those items of data. The new information expressed in  $\mathbf{D}$  comes from experiments.

In the present room-acoustic example, the quantity  $p(\Theta)$  is a probability density function (PDF) that represents one's initial knowledge (degree of belief) about the parameter values before taking into account the experimental data  $\mathbf{D}$ . It is therefore known as the prior probability for  $\Theta$ . The term  $p(\mathbf{D} | \Theta)$  reads “the probability of the data given the parameter  $\Theta$ ” and represents the degree of belief that the measured data  $\mathbf{D}$  would have been generated for a given value of  $\Theta$ . It represents, after observing the data, the likelihood of obtain-

ing the realization observed as a function of the parameter  $\Theta$ . It is often called the likelihood function. This likelihood represents the probability of fitting the measured data  $\mathbf{D}$  to the hypothesis realized by the parameter  $\Theta$  via model  $\mathbf{H}(\Theta)$ . A straightforward way to quantify this fitting is to build a difference function between the modeled realization  $\mathbf{H}(\Theta)$  and the experimental data  $\mathbf{D}$ . The difference function is also called residual error. In other words, the likelihood function is simply the probability of the residual error.

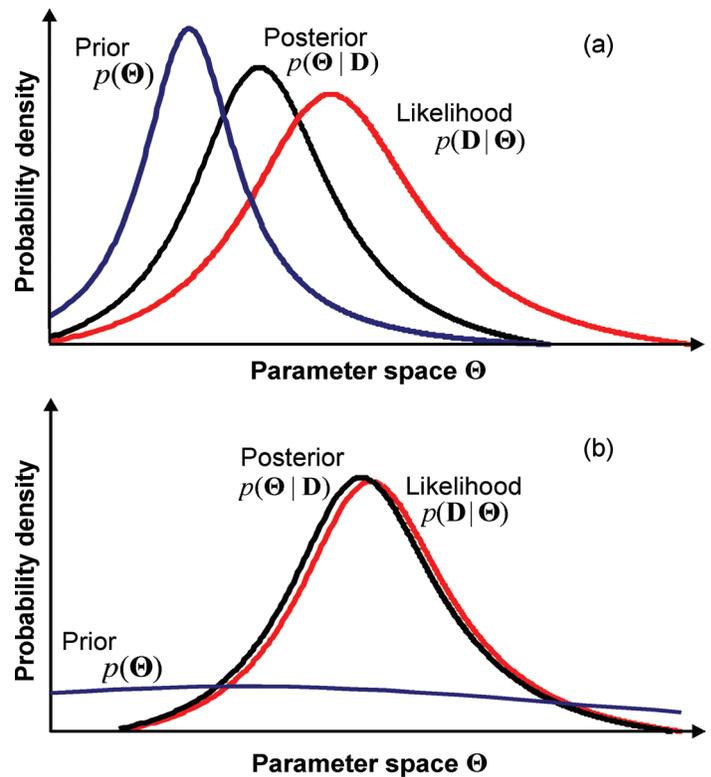
The term  $p(\Theta|\mathbf{D})$  represents the probability of parameter  $\Theta$  given the data  $\mathbf{D}$  after taking the data into account. It is therefore called the posterior probability. The quantity  $p(\mathbf{D})$  on the left-hand side of Equation 2 represents the probability that the data are observed. After the data measurement, it is a normalization constant and ensures that the posterior probability (density function) integrates to unity.



**Figure 2.** Model-based parameter estimation scheme is an inverse problem. Based on a set of experimental data and a model (hypothesis), the task is to estimate the parameters encapsulated in the model. In general, the experimenter knows that an appropriate set of parameters will generate a hypothesized “data” realization that approximates the experimental data.

Acousticians typically face data analysis tasks from experimental observations, and they are challenged to estimate a set of parameters encapsulated in the model  $\mathbf{H}(\Theta)$ , also called the hypothesis. Figure 2 shows this task schematically. The relevant background information is that the experimenter knows that an appropriate set of parameters will generate a hypothesized “data” realization via a well-established prediction model (hypothesis) that approximates the experimental data well.

Figure 3 interprets Bayes’ theorem as presented in Equation 2. The posterior probability, up to a normalization constant,  $p(\mathbf{D})$ , is calculated by updating the prior probability,  $p(\Theta)$ . Once the experimental data  $\mathbf{D}$  become available, the likelihood function  $p(\mathbf{D}|\Theta)$  is a multiplicative factor that modifies the prior probability so as to become the posterior probability (up to a normalization constant). Bayes’ theorem



**Figure 3.** Bayes’ theorem (in Equation 2) used to estimate a parameter with the same dataset as expressed in the likelihood function  $p(\mathbf{D}|\Theta)$ , yet different prior probabilities  $[p(\Theta); \text{density functions}]$  affect the posterior probability  $[p(\Theta|\mathbf{D}); \text{density function}]$  differently (Cowan 2007). (a) A more sharply peaked prior  $p(\Theta)$  has a stronger influence on the posterior  $p(\Theta|\mathbf{D})$ . (b) A flat prior  $p(\Theta)$  has almost no influence on the posterior  $p(\Theta|\mathbf{D})$ . Objective Bayesian analysis often involves a broad or flat prior, a so-called noninformative prior that represents maximal prior ignorance, therefore allowing the experimental data to drive the Bayesian analysis.

represents how our initial belief, or our prior knowledge, is updated in the light of the data. This corresponds intuitively to the process of scientific exploration: acoustical scientists seek to gain new knowledge from acoustic experiments.

In Equation 2, Bayes’ theorem requires two probabilities to calculate the posterior probability up to a normalization constant. These are the prior probability and the likelihood function. Use of the prior probability in the data analysis was an element in the controversy between the frequentist and Bayesian schools. Frequentists have directed criticism at the “subjective” aspect of the prior probability involved in the Bayesian methodology. Indeed, different prior assignments will lead to different posteriors using Bayes’ theorem (see Figure 3; Cowan, 2007). Figure 3a demonstrates that if the prior probability is sharply peaked in a certain range in the parameter space, the likelihood function multiplied by the prior probability may lead to a posterior probability peaked at a significantly different position than indicated by either the prior probability or likelihood function. Sharply

peaked probability density functions encode a high degree of belief in the truth of the parameter values lying at certain (narrow) ranges in the parameter space. Assigning the prior probability in this way implies injection of subjective belief into the data analysis. In that case, one already knows the parameters accurately so there is no need for experiment unless the experiment can yield a sufficient quality and quantity of data so that the likelihood function, and hence the posterior probability, become even more sharply peaked with respect to the parameter values.

The use of prior probability is actually a strength, not a weakness, of Bayesian analysis because in the extreme case that we know of, the parameters of the prior probability are all heaped at a single value so that the prior probability is zero elsewhere. Bayes' theorem immediately shows that this feature carries through to the posterior, in accord with intuition but not with frequentist methods of data analysis. The assignment of prior probability should proceed from the prior information. If it is not known how to assign a prior distribution from the prior information and often this is the case where limited prior knowledge on the parameter values is available, then it is safe to assign a broad prior probability density as shown in **Figure 3b**. Below, the objective Bayesian analysis proceeds by assigning such a noninformative prior probability.

### Maximum Entropy Prior Probabilities

Bayes' theorem involves both sources of information: the prior information about the process (parameters) under investigation and the experimental data observed in the experiment through the likelihood function. The prior probability encodes the experimenter's initial degree of belief in the possible values of parameters, and the likelihood function encodes the degree of belief in the possible fit (or misfit) of the hypothesized prediction generated by the parameters to the experimental data. Both probabilities must be assigned (yet not assumed) to apply Bayes' theorem.

Berger (2006) rebutted a common criticism of the Bayesian school arising from the supposedly subjective use of prior probabilities. Even the first Bayesians, including Bayes (1763) and Laplace (1812), performed probabilistic analysis using constant prior probability for unknown parameters. A well-known technique that is often used in objective Bayesian analysis relies on whatever is already known about a probability distribution to assign it; this is the maximum entropy method and generates so-called maximum entropy

priors (Jaynes, 2003). Jaynes (1968) applied a continuum version of the Shannon (information theoretic) entropy, a measure of uncertainty, to encode the available information into a probability assignment. In the objective Bayesian literature (Jaynes, 1968; Gregory, 2010), this is termed the principle of maximum entropy, and it provides a consistent and rigorous way to encode testable information into a unique probability distribution. The principle of maximum entropy assigns a probability,  $p(\cdot)$ , that maximizes the entropy,  $S[p(\cdot)]$ , involving distributing the probability as noncommittally as possible while satisfying all known constraints on the distribution. The resulting distribution is also guaranteed to be nonnegative.

### Prior Probability Assignment

To assign the prior probability (in **Equation 2**) objectively, no possible value of a parameter should be privileged over any other, except to the extent necessary to conform to any known constraints on the distribution. A universal constraint is normalization such that the prior probability density integrates to unity. The principle of maximum entropy, as its name suggests, assigns the density,  $p(\cdot)$ , by maximizing its entropy,  $S[p(\cdot)]$ , subject to this constraint and any other. In the absence of further constraints, the result is a constant-value probability density bounded by a certain wide parameter range, so-called uniform prior probability (Jaynes, 1968; Gregory, 2010).

### Likelihood Function Assignment

The likelihood function represents the probability of the residual errors. In the example shown in **Figure 1**, these errors are essentially the differences between the hypothesized prediction (solid line) and the experimental data (black dots). When assigning the likelihood function, one should incorporate only what is known about the residual errors. In other words, objective Bayesian analysis should not implicitly commit itself to any information that is not known to the experimenter.

As ever, this probability distribution must integrate to unity. Also, in many data analysis tasks, the experimenters know in advance that the model (such as **Equation 1** in the example mentioned above) is capable of representing the data well, so that the residual errors should feature finite values and the variance is formally noninfinite. Taking into account the finite variance in a maximum entropy procedure on a continuous space of uniform measure, the result is the Gaussian or normal probability distribution for the residual errors

(Gregory, 2010). To be precise, this Gaussian assignment is different from assuming the statistics of the residual errors to be Gaussian. It is the consequence of little information on the finite, yet unspecified, residual error variance being available.

In some other data analysis tasks, the experimenter knows only that the model represents the data well enough for the residual errors to have a noninfinite mean value. Taking account of this finite mean in the maximum entropy procedure gives rise to an exponential distribution; in this case, the parameter space must not be unbounded above and below. (When the mean and variance are both noninfinite, the result is a Gaussian distribution with nonzero mean and finite variance.)

In summary, maximization of the Shannon-Jaynes entropy (Gregory, 2010) is a wholly objective procedure in the data analysis. The resulting distribution is guaranteed to show no bias for which there is no prior evidence.

### Two Levels of Bayesian Inference

In many acoustic experiments, there are a finite number of competing models (hypotheses),  $H_1, H_2, \dots, H_M$ , that are in competition to explain the data. In the room-acoustic example above,  $H_1$  is specified in Equation 1 as containing one exponential decay term and one noise term, but the same data in Figure 1 may also be alternatively described by  $H_2$  containing two exponential decay terms (double-rate decay) with two further parameters. Only one of the models is expected to explain the data well. Figure 4 illustrates this scenario. In practice, architectural acousticians often expect single-, double-, or triple-rate energy decays (Xiang et al., 2011; Jasa and Xiang, 2012). In the case of plausible competing models (e.g., double rate, triple rate), it would be unhelpful to apply an inappropriate model to the parameter estimation problem (Xiang et al., 2011). Before undertaking parameter estimation, one should ask, “Given the experimental data and alternative models, which model is preferred by the data?”

Bayesian data analysis applied to solving parameter estimation problems, as in the example above, is referred to as the first level of inference, whereas solving model selection problems is known as the second level of inference. Bayesian data analysis is capable of performing both the parameter estimation and the model selection by use of Bayes’ theorem. We begin below with the second level of inference, namely model selection. This top-down approach is logical;

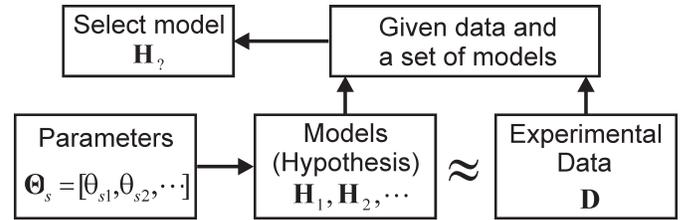


Figure 4. Second level of inference, model selection, is an inverse problem about selecting one of a finite set of models. Based on the experimental data and a finite set of models (hypotheses), the task is to infer which model is preferred by the data.

one should determine which of the competing models is appropriate before the parameters appearing in the model are estimated (Xiang, 2015).

### Model Selection: The Second Level of Inference

Within a set of competing models, the more complex models (generally those with more estimable parameters) will always fit the data better. But models with excessive numbers of parameters often generalize poorly (Jefferys and Berger, 1992). To penalize overparameterization, Bayes’ theorem is, accordingly, applied to one of the competing models,  $H_s$ , among a finite model set,  $H_1, H_2, \dots, H_M$ , given the data  $D$ , deferring any interest in model parameter values. We simply replace  $\Theta$  in Equation 2 by the model  $H_s$ . In this application of Bayes’ theorem, the likelihood function,  $p(D|H_s)$ , is referred to as the (Bayesian) evidence. The posterior probability of the model  $H_s$  given the data  $D$  is simply proportional to the evidence because the principle of maximum entropy assigns identical prior probability to each model.

Although more complex models may fit the data better, they pay a penalty by strewing some of the prior probability for their parameters where the data subsequently indicate that those parameters are extremely unlikely to be. There is, therefore, a trade-off between goodness of fit and simplicity of model, and this can be seen as a quantitative generalization of the qualitative principle known as Occam’s razor, which is to prefer the simpler theory that fits the facts (Jefferys and Berger, 1992).

The model selection process calculates the posterior probability of each of the finite number of models to determine which model has the greatest posterior probability (or after an increasing trend, no more significant increases will be observed; Jeffreys, 1965) and, if desired, to rank the competing models.

## Parameter Estimation: The First Level of Inference

Once a model has been selected according to the experimental data, denoted as model  $H_s$ , then the Bayesian framework is available to estimate its parameters  $\Theta_s$ . We now write the model with its parameters explicit as  $H_s(\Theta_s)$ ; for instance, model  $H_1(\Theta_1)$  in Equation 1 contains one exponential decay term with its parameters collectively denoted as  $\Theta_1 = \{\theta_0, \theta_1, \theta_2\}$ . Bayes' theorem is now applied as before so as to estimate the parameters  $\Theta_s$ , given the model  $H_s$  and the data  $D$ . The quantity  $p(D)$  in Equation 2 now becomes  $p(D|H_s)$ , the probability of the data  $D$  given the model  $H_s$ .

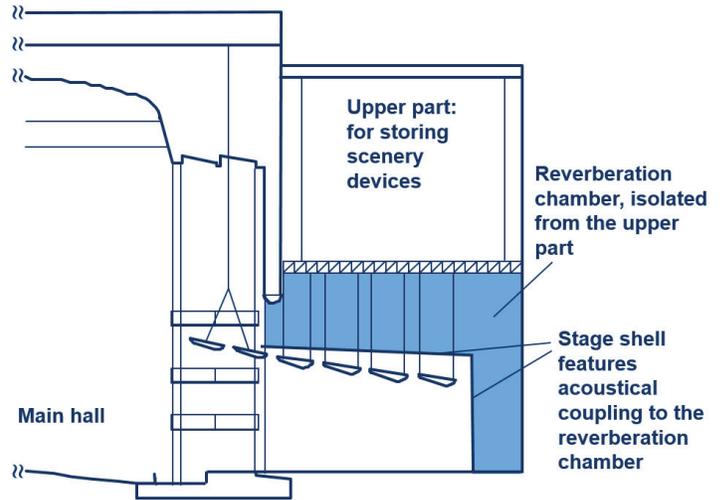
## Two Levels of Inference in One Unified Framework

Although the quantity  $p(D|H_s)$  plays the role of a normalization constant at the first level of inference, it is identical to the Bayesian evidence, which is central to model selection, the second level of inference. Bayes' theorem can be expressed as

$$\text{posterior} \times \text{evidence} = \text{prior} \times \text{likelihood} \quad (3)$$

On the right-hand side of Equation 3, the prior probability and the likelihood function of parameters are inputs, whereas the posterior probability and the (Bayesian) evidence are the outputs of Bayesian data analysis. The form of the posterior is determined as the prior multiplied by the likelihood and the evidence is its normalization constant. The posterior probability is the output in the first level of inference, namely, parameter estimation, whereas the evidence is the output for the second level of inference, namely, model selection (Sivia and Skilling, 2006).

To calculate the evidence, the likelihood function multiplied by the prior probability must be explored (integrated) over the whole parameter space (Xiang, 2015). That demands substantial computational effort. Often it is necessary to use numerical sampling methods based on the Markov Chain Monte Carlo approach. In fact, Equation 3 indicates implicitly that model selection and parameter estimation can both be accomplished within a unified Bayesian framework. Once sufficient exploration has been performed to determine the evidence, the explored likelihood function multiplied by the prior probability can contribute to the normalized posterior probability because the evidence is also estimated. From estimates of the posterior distributions, it is straightforward to



**Figure 5.** Stage house of opera/theatre venues features a stage shell (concert hall shaper) that is readily deployable to convert the opera/ballet setting to symphonic performance setting (Jaffe, 2010). For symphonic purposes, the stage shell features acoustical coupling between the reverberant stage house behind and above the shell with the main audience floor.

estimate the mean values of the relevant parameters, their uncertainties in terms of associated individual variances, and the relationships between the parameters of interest (Xiang, 2015).

## Applications in Acoustics

Both levels of Bayesian inference have been applied in acoustics. This section briefly sets out some recent examples including applications in room acoustics, communication acoustics, and ocean acoustics. These applications all have in common the existence of well-established, or at least well-understood, models for predicting the processes under investigation, yet only a set of experimental data is available to study the process.

## Multiple-Rate Room-Acoustic Decay Analysis

A number of recently designed concert halls have implemented reverberation chambers coupled to the main floor so as to create multiple-rate energy decays. A further application is the design and adaptation of a stage shell system, that is, a “concert hall shaper” in opera/theatre venues to couple with reverberant stage houses as shown in Figure 5 so as to attain two desirable yet competing auditory features: clarity and reverberance (Jaffe, 2010; Xiang et al., 2011). Architectural acousticians are often challenged to quantify reverberation characteristics in (connected) coupled spaces (Jasa and Xiang, 2012). On extending the model used in the

example above, a predictive model for sound energy decay functions consists of a number of exponential decay terms and one noise term. Models with one, two, or three rates of decay can be denoted as  $H_1, H_2, H_3$ , in which the multiple decay times,  $T_1, T_2$ , or  $T_3$ , via decay constants are of practical concern.

This room-acoustic problem is readily soluble using the two levels of Bayesian inference (Xiang, 2015). In **Figure 4**, the data analysis begins with a set of data points (in the time domain) and a set of models with unspecified number of exponential terms. Given the experimental (sound energy decay) data, architectural acousticians need the answer to the higher level question of how many exponential terms there are in the energy decay data before estimating detailed decay parameters as indicated in **Figure 2**.

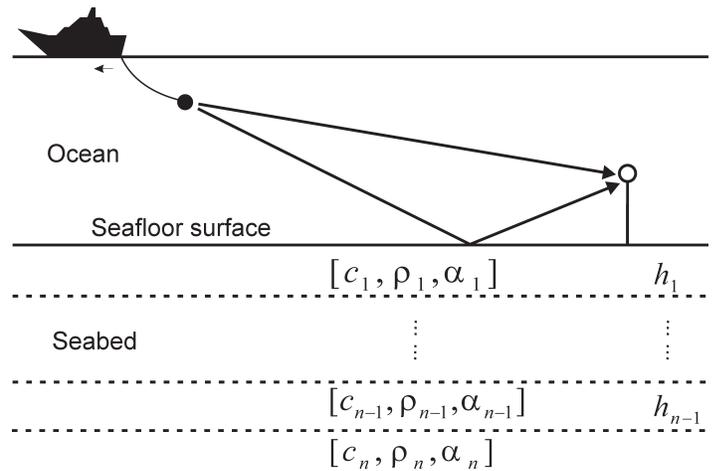
### Sparse Sensors for Direction of Arrival Estimation

Sound source localization using a two-microphone array in, e.g., communication acoustics is a cost-efficient yet challenging technique because just two omnidirectional microphones set a known distance apart from each other (but not binaural configurations) provide incomplete information. Escolano et al. (2014) reported an application using generalized cross-correlation (GCC) between the two microphone signals with a dedicated phase transformation (GCC-PHAT) as a prediction model for the direction of arrivals. In this application, a complete solution embodies two levels of inference, model selection to estimate the number of concurrent sound sources in a noisy, reverberant environment and parameter estimation to determine the direction of arrivals. The GCC-PHAT model-based processing of two microphone signals involves the predictive models  $H_1, H_2, H_3, \dots$ , with one, two, or three sources and so on. Yet there is no need to impose the number of concurrent sources a priori because it is sufficient to apply Bayesian model selection so as to estimate the number of concurrent sources. Once the model is selected, given the two microphone signals, to contain the selected number of concurrent sources, it is possible to estimate their angular parameters (direction of arrivals).

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**Bayes' theorem represents how our initial belief, or our prior knowledge, is updated in the light of the data.**

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**Figure 6.** Geoacoustic transdimensional inversion for seabed structure.  $h_p$ , Layer thickness;  $c_p$ , sound speed;  $\rho_p$ , density;  $\alpha_p$ , attenuation. From Dettmer et al. (2010) and Dosso et al. (2014).

### Geoacoustic Transdimensional Inversion of Seabed Properties

Acoustics is widely used to study the seabed. **Figure 6** shows an at-sea experiment (Dettmer et al., 2010; Dosso et al., 2014) in which a ship tows an impulsive acoustic source past a moored receiver that records reflections off the seafloor and subbottom layers; the data are processed as reflection coefficients as a function of incident angle and frequency. The aim is to estimate a geoacoustic model of the seabed including an unknown number of layers, with unknown layer thickness  $h_i$ , sound speed  $c_i$ , density  $\rho_i$ , and attenuation  $\alpha_i$  in each layer. The two levels of inference are carried out simultaneously using transdimensional inversion to sample probabilistically over models with differing numbers of layers. The result is a set of depth-dependent probability profiles for the geoacoustic parameters  $(c, \rho, \alpha)$  averaged over the number of layers, with the advantage that the uncertainty in the number of layers is included in the parameter uncertainty estimates.

### Concluding Remarks

Bayesian probabilistic analysis has recently been applied to an increasing extent in acoustic science and engineering. Objective Bayesian analysis ensures that the output of Bayes' theorem is a posterior probability distribution based precisely on the information put into it, not more and not less, and that the information has been utilized systematically and objectively. Many data analysis tasks in acoustics embody two levels of inference, namely, model selection and, within the chosen model, parameter estimation. Objective Bayesian analysis provides solutions of both levels using Bayes' theorem within a unified framework.

## Acknowledgments

The authors are grateful to Dr. Mingsian Bai for stimulating discussions and to Dr. Stan Dosso for details of the geoacoustic transdimensional inversion.

## Biosketches



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**Cameron J. Fackler**, obtained his BS degree in 2010 in Applied Physics, and MS degree in 2011 and PhD degree in 2014 in Architectural Sciences with Professor Ning Xiang at Rensselaer Polytechnic Institute. During his doctoral research, he applied Bayesian analysis to design microperforated panel absorbers and to analyze

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