WAVE PROPAGATION IN ARRAYS OF SCATTERERS TUTORIAL: PART 1

Julian D. Maynard
Department of Physics, The Pennsylvania State University
University Park, Pennsylvania 16802

When sound waves, propagating in a uniform medium, encounter a foreign object, the sound is scattered in all directions. When the size of the object is about the same size as the wavelength of the sound, the scattering may be called "strong" and its pattern may be complicated. If the geometry of the object is relatively simple, it would be possible to calculate the scattered sound field—such a situation is represented by Fig. 1a. However, if there is a second object, located a few wavelengths away from the first object, then the scattered sound may reflect back and forth between the objects, undergoing "multiple scattering," as shown in Fig. 1b. The multiple scattering makes any calculation far more difficult, and the problem would rapidly grow in difficulty if several more scattering objects were added. What then if a medium were filled with such scatterers, as illustrated in Fig. 2? One might imagine that such a situation would be hopeless. However, if the scatterers are arranged periodically, like the atoms in a crystal, then a sound field may be readily calculated to high accuracy.[1] Furthermore, if the scatterers were in a disordered configuration, one could make significant qualitative and quantitative predictions as to how sound would behave if it encountered such a system.[2] There is also a possible configuration of scatterers intermediate between periodic and disordered, referred to as quasicrystalline,[3] for which an accurate understanding of the behavior of the sound is again possible. This article is an introduction to understanding sound propagation in periodic arrays of scatterers. Later articles will treat the cases of disordered and quasicrystalline arrays of scatterers.

"If scatterers are arranged periodically like atoms in a crystal, then a sound field may be readily calculated to high accuracy."

Periodic array of scatterers

The importance of understanding wave propagation in a periodic array of scatterers is illustrated in Fig. 3. Consider a plate (e.g., a floorboard) with a source of vibration at one end and some listeners at the far end. The source generates transverse (flexural) waves, and these waves propagate to the far end of the plate, radiate sound and create an annoyance for the listeners (Fig. 3a). Usually, for structural reasons, a plate will have a rib on it, and that rib reflects the vibration (as shown in Fig. 3b) so there is less vibration transmitted, and less noise at the other end of the plate. If one rib reflects the vibration and reduces the annoyance, why not a series of many ribs? For ease of manufacture, the ribs may be identical and arranged periodically, as illustrated in Fig. 3c. One might expect that the array of ribs would greatly decrease the vibration, with transmission proportional to some large power of the single-rib transmission coefficient. However, this is not at all what happens. The remarkable result for the periodic array is that the first rib reflects the vibration, but all the rest transmit without any further reduction, at least within ranges of frequencies, called "pass bands."

This remarkable property of periodic arrays of scatterers is well known in solid state physics,[4] where a typical application would be the understanding of the electrical conductivity of a metallic crystal. In a crystalline wire consisting of moving electrons and fixed positive ions, as illustrated in Fig. 4, a classical electron would be very strongly scattered by the ions as it tried to move down the wire, and one would have to conclude that a metal should be a very poor conductor of electricity. The fact that metals are actually good conductors

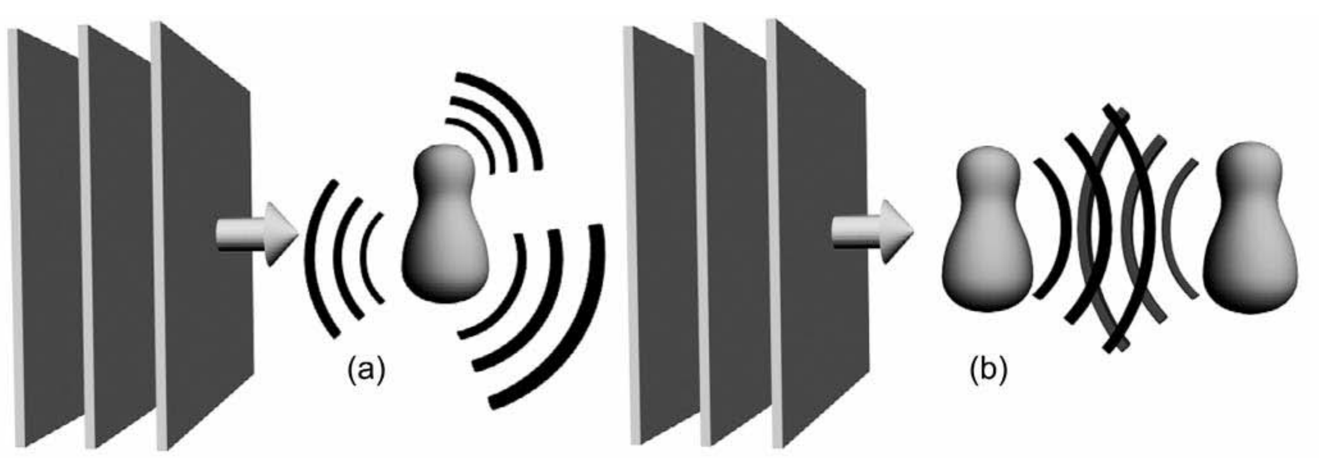

Fig. 1. Illustrations of sound scattering from relatively simple objects. (a) Single object. (b) Multiple scattering between two objects greatly complicates the scattering problem.

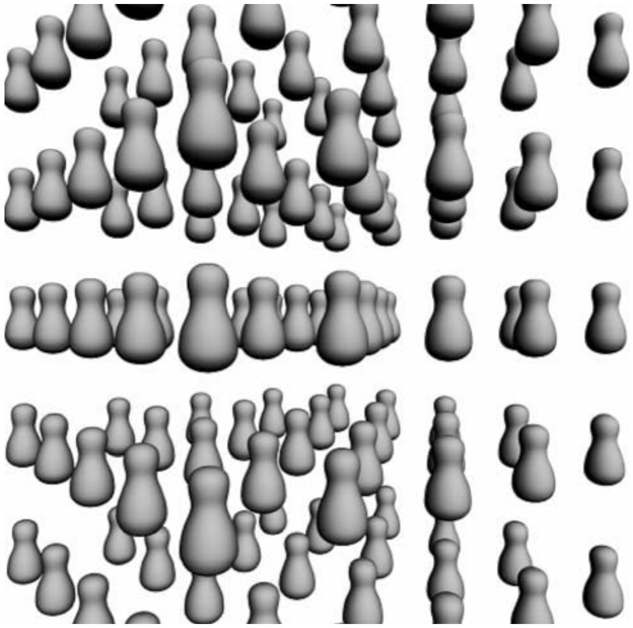

Fig. 2. Illustration of a system where the medium is filled with scatterers.

of electricity is due to two features: (a) because of quantum mechanics, the electron behaves as if it were a *wave*, and (b) in a crystal the ions are arranged *periodically*.

A one-dimensional acoustic array

The properties of soundwaves (or electrons) in periodic arrays of scatterers may be thought of as resulting from some special coherence in the multiple scattering within the periodic system of scatterers. These properties have a rigorous mathematical foundation in Floquet's theorem[5] that is generalized in physics and called Bloch's theorem.[6] Rather than use a highly mathematical approach, we will explain the behavior of wave propagation in a periodic array of scatterers with a simpler, graphical presentation.

To understand how sound propagates in a periodic system, we will study a one-dimensional periodic system because the one-dimensional approach may be readily applied to three orthogonal dimensions. First it will be necessary to establish a model and some notation. The one-dimensional model wave medium will be a string, having some mass per unit length ρ_L, stretched to a constant tension. The sound waves will be transverse vibrations of the string (i.e., the displacement of the string will be orthogonal to the length of the string), traveling with a wave speed *c*. It will be assumed that the string is infinitely long, so that boundary conditions may be ignored. The periodic array of scatterers will be point masses positioned at equal intervals (with spacing "*a*," called the "lattice constant,") along the string. Each point scatterer will have the same mass, *m*. A section of the string with a mass that is repeated periodically will be referred to as a "unit cell." The time dependence of the motion of the string will be simple harmonic, given by $\cos(2\pi ft)$, where *f* is the frequency of the transverse vibrations. With the string extending along the *x*-axis, the displacement of the transverse vibration in the *y*-direction, ψ, will be a function of its *x*-position, thus $\psi(x)$. It should be noted that this formulation includes standing waves only; traveling waves may be formed as linear combinations of standing waves.[7]

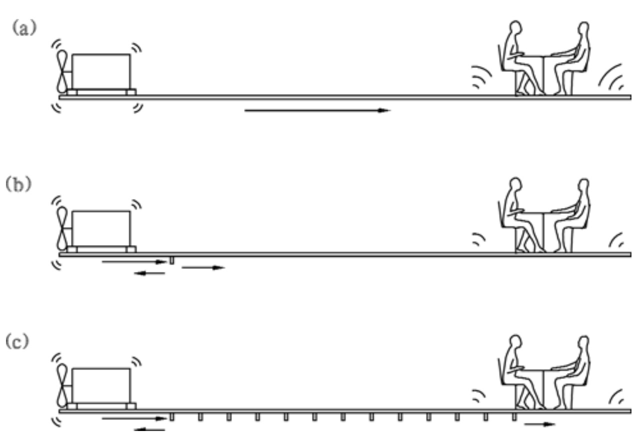

Fig. 3. An illustration of the importance of understanding the vibration of a system with a periodic array of scatterers. (a) Waves propagating down a plate may make noise at a far end. (b) A rib would reflect the wave, reducing the noise. (c) Remarkably, more ribs, if arranged periodically, would not further reduce the noise.

Fig. 4. The properties of waves in periodic arrays of scatterers are well known in solid state physics. In this case the waves are the quantum mechanical nature of the electron and the scatterers are ions in a crystal lattice.

The treatment of the model system involves finding the possible modes of vibration of the string and the natural frequency for each mode, for a given value of the "strength" of the scatterers (that is, the size of the mass *m*). The modes and frequencies may be indexed with subscripts.

Zero masses

The first possibility to consider is what happens when the mass of each scatterer is zero (i.e., $m=0$). That is the situation for which the string actually has no scatterers at all. In this case, the modes of vibration are proportional to $\cos(2\pi x/\lambda)$ or $\sin(2\pi x/\lambda)$, where $\lambda= c/f$ is the wavelength of the vibration. Because the string is infinite in length, it is not possible to show the mode of vibration for the entire string in a figure. Because the modes are periodic in the position *x*, with periodicity λ, figures of modes may be made by illustrating them only over a finite length *l* (from $x = 0$ to $x = l$), with *l* at least as large as a half-wavelength. It is also not possible to show all of the modes for all possible (i.e., arbitrary) wavelengths λ; however, it will turn out that only certain wavelengths will be important for understanding what happens when the scatterers (with *m* greater than 0) are placed on the string. Thus, the following conditions are added: (a) examples of the modes of vibration will be limited to those modes that correspond to fitting an integer number of half-wavelengths in the length *l* (so that λ / 2*l* = integer) and (b) when scatterers are added, they will be placed with a lattice constant *a* such that an integer number of lattice constants will fit in the length *l* (*l* / *a* = integer). These last relationships constitute forming "periodic boundary conditions" applied to the system at $x=0$ and $x=2l$. It should be noted that with these conditions, the frequencies of the modes (for the case

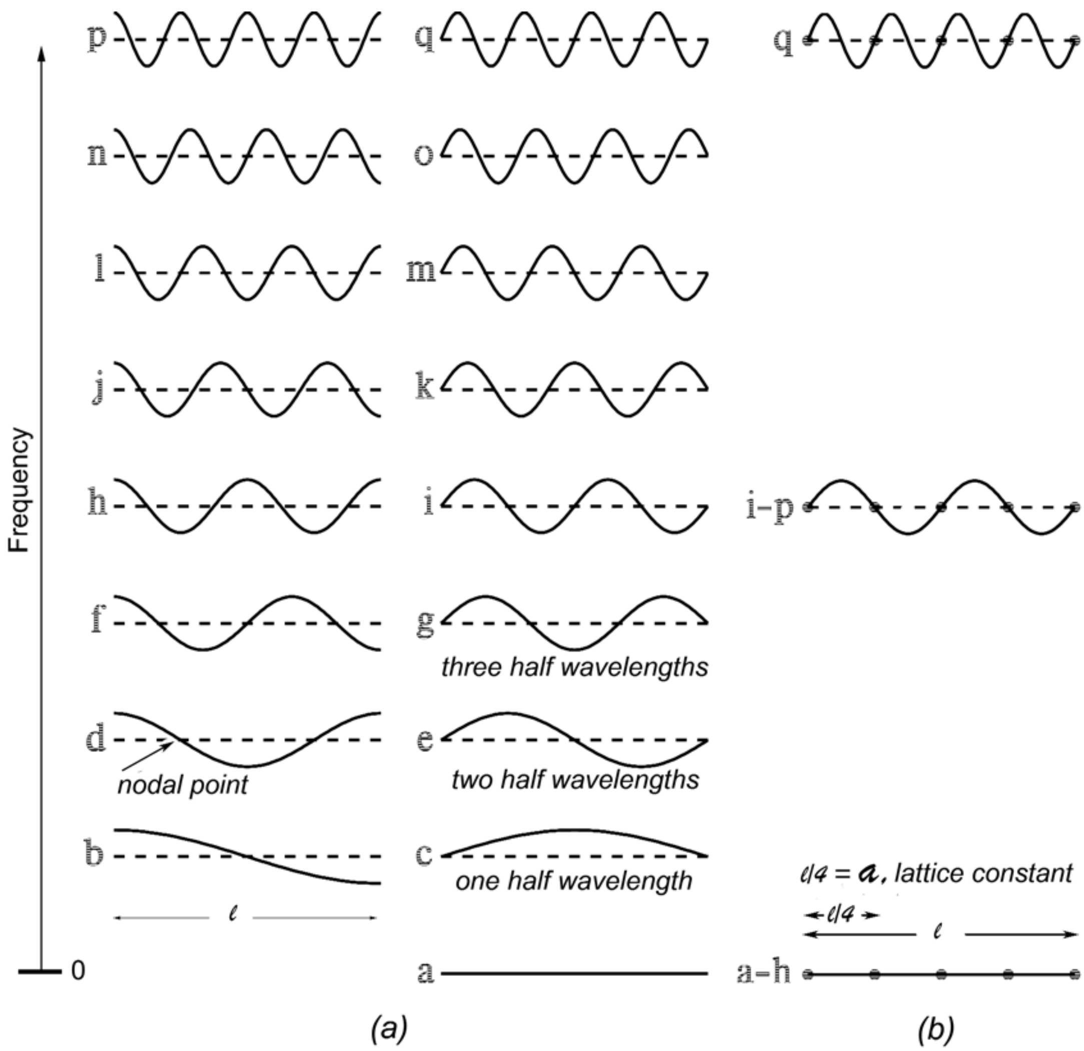

Fig. 5. (a) The modes of vibration for a string with zero scatterers. The zero frequency equilibrium position of the string is shown as mode "a." The vertical position of the dashed line for the different modes is proportional to the frequency of the mode; the harmonic sequence of frequencies is evident. Each frequency has two "degenerate" modes, corresponding to cos(2πx/λ) and sin(2πx/λ). (b) The result of placing point mass scatterers on the string, dividing the length shown into four equal parts, with the mass of the scatterers being infinite. Some of the modes are "suppressed" down to the equilibrium state "a" from Fig. 5 (a). Some of the modes from Fig. 5 (a) are unchanged, because the infinite masses wind up at nodal points, as for modes "i" and "q." Other modes have their modes "pulled" so as to become equivalent to "i," "q," etc.

with the string free of scatterers) form a harmonic sequence, given by $f = \nu c/2l.$, $\nu = 0, 1, 2, \cdots$. The zero frequency given by $\nu=0$ corresponds to the equilibrium position of the string, with no transverse displacement. In our treatment, this zero frequency equilibrium position will actually be counted as a mode.

With the notation and conditions established above, the first seventeen relevant modes of vibration of a string that is free of scatterers is shown in Fig. 5a. The letters labeling the different modes will be used for later reference. The zero frequency equilibrium state of the string with no transverse displacement is shown as mode "a." For each mode, the dashed line indicates the equilibrium position of the string. The positions where the string crosses the dashed line are called "nodal points," that for standing waves *never* move. The figure is drawn so that the vertical position of the dashed line of each mode is proportional to the frequency of the mode. For the plain string, the frequencies are harmonic, so the modes are drawn in the figure with equal vertical spacing. At each vertical level (or frequency) there are two modes given by $\cos(2\pi x/\lambda)$ and $\sin(2\pi x/\lambda)$ that have the same frequency. Modes that have the same frequency are said to be "degenerate." In aid of understanding what happens when scatterers are added, one should take particular note of modes "i" and "q" in Fig. 5a.

Infinite masses

The next step in understanding the effect of periodic scatterers is to jump from the extreme case of the string with no scatterers (m=0), to the opposite extreme case with the string having point mass scatterers, where the masses are infinite, that is $m = \infty$. With reference to Fig. 5b, infinite masses will be placed on the string with a lattice constant $a = l / 4$, beginning with the leftmost end of the section of string. In our example, five infinite masses will be placed on the string, dividing it into four equal segments as shown in Fig. 5b. Because an infinite mass cannot be moved, any mode of the string that "tries" to move at the site of an infinite mass will be completely suppressed. Thus, with reference to the labeling in Fig. 5a, modes "b" through "h" are suppressed (cannot exist), and are pushed down to the equilibrium mode "a." However, some of the modes can exist and survive unchanged, because when the infinite masses are placed on the string, they wind up at the nodal points of the mode. Because the nodal points never move anyway, placing infinite masses there has no effect. Such a situation occurs for modes "i" and "q" in Fig. 5a and are shown in Fig. 5b. These modes have nodal points that divide the length of string shown into four parts, *exactly* where the infinite masses are placed. For modes "j" through "p," the effect of the infinite masses is to "pull" the nodal points so that the transverse displacement looks just like mode "i," and they become equivalent to that mode. Mode "q" also picks up duplicate modes, but these modes (from above "q") were not included in Fig. 5a. The result of placing scatterers with infinite mass, dividing the length *l* into four parts, is illustrated in Fig. 5b. Apart from the equilibrium mode "a," only two distinct modes survive from Fig. 5a. However, it is important to note that there are still an infinite number of modes above mode "q" that correspond to fitting an integer number of half-wavelengths inside the lattice constant, *a*.

A bit of an explanation is required here. We have used the expressions "suppressed" and "pulled" to describe what is happening to the modes when the masses are abruptly changed from zero to infinity. This is not whimsy. It clearly describes the effect on the modes when the masses are slowly increased from zero to infinity, as will be described next.

Finite masses

If we mathematically derive and plot the frequencies of the seventeen modes as a function of increasing mass from zero to infinity, we obtain the graph illustrated in Fig. 6. Because $m=\infty$ cannot fit on a finite size figure, we can use an artifact to make the plot fit. We will let the variation of the mass be reflected in a parameter given by $\arctan(m/\rho_L a)$. This parameter varies from zero to $\pi/2$ as the mass varies from zero to infinity. The dimensionless combination $m/\rho_L a$ gives the mass of the scatterer relative to the mass of the string between the scatterers because ρ_L is the mass per unit length and a is the length of a unit cell. A plot indicating how the frequencies of the modes (the vertical positions of the modes in Fig. 5a and Fig. 5b) change as the mass goes from zero to infinity is shown in Fig. 6.

On the far left of Fig. 6, where $m=0$, the harmonic sequence of the seventeen frequencies of the string without any masses (Fig. 5a) is evident. The letters on the left indicate the modes that have these frequencies (the modes are two-fold degenerate except for "a"). Note that the harmonic sequence of frequencies continues upward to infinity, with the sequence generated by fitting an integral number of half-wavelengths into the length *l*. The far right of Fig. 6 shows some frequencies when the mass of the scatterers is infinite as is seen in Fig. 5b. These are the first three frequencies of a sequence that also continues upward to infinity, but in this case the sequence is generated by fitting an integral number of half-wavelengths inside the length *a*, rather than inside the length *l*. The letters on the left of Fig. 5b indicate the modes associated with the frequencies on the far right of Fig. 6. The

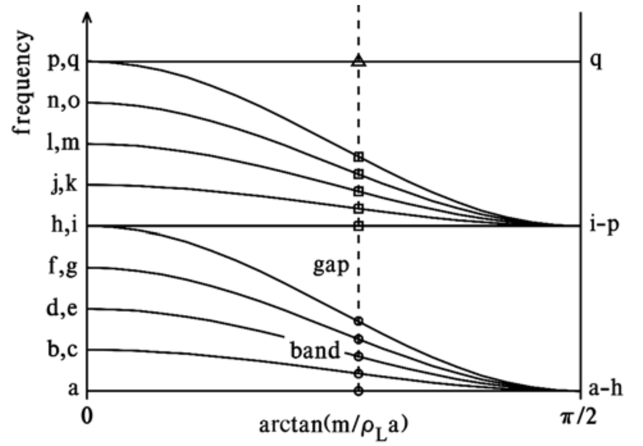

Fig. 6. A plot indicating how the frequencies of modes vary as the mass of the scatterer in a periodic array goes from zero to infinity. The letters refer to the labels of Figs. 5 (a) and (b). The frequency levels on the left are the equally spaced harmonics of the string without scatterers, and the frequency levels on the right are the natural frequencies of the "local oscillators" formed between the infinite masses.

continuous lines connecting the frequencies on the far left with the frequencies on the far right indicate how sets of frequencies evolve as the mass of the scatterers is continuously varied from zero to infinity. The connecting lines consist of sets of *four* curved lines lying in the regions located between pairs of horizontal *straight* lines. One can see that, for example, lines "b," "c," "d," "e," "f," "g," and "h" in Fig. 6 are lower in frequency when the masses are greater than zero and indeed, are "pulled" toward line "a" as described.

The vertical dashed line in Fig. 6 indicates an example when the mass of the scatterers lies somewhere between zero and infinity; the frequencies are given by the symbols (circles, squares, and triangles) at the intersections of the dashed line with the curved lines. The frequencies of each mode with a non-zero mass is lower than the same mode with a zero-mass. The collection of frequencies given by the symbols on one horizontal line and the curved lines just above it (and below the next horizontal line) is referred to as a "pass band," or simply "band." In Fig. 6, the circles constitute one band, and the squares constitute a second band; this continues upward to infinity. The empty space between the uppermost symbol on a curved line in a band and the lower symbol on the horizontal line in the next band is referred to as a "stop band" or "gap." The pattern of frequencies, with bands and gaps, along the dashed line, continued up to infinity, is referred to as "band structure;" note that the pattern of frequencies is not harmonic. It should be kept in mind that Fig. 6 is for the particular case where $l = 4a$. For the general case where $l = Na$, with N an integer, the number of frequencies in one band would be $N+1$. If the lattice constant a is fixed, then as l becomes large (and N becomes large), then the frequencies in a band get close together, and the band approaches a continuous distribution of frequencies.

Another way of describing the evolution of the frequencies as a function of the scatterer mass is to begin with the case where the mass is infinite, and then consider what happens when the mass is decreased. This corresponds to starting at the far right side of Fig. 6 and moving to the left. This description will also give rise to another, far-reaching application of scattering. When the masses are infinite, then the motion of a length of string between two masses can have no effect on any other section of string—thus the infinite masses divide the string into "isolated local oscillators." Each local oscillator consists of a string of length *a*, clamped at both ends. The normal modes of each local oscillator consist of fitting an integer number of half-wavelengths into the length *a*, and the natural frequencies form a harmonic sequence with $f_\nu = \nu c/2a$. The frequencies with ν equal to 0, 1 and 2 are the ones on the right in Fig. 6, and the sequence continues upward to infinity. When isolated with the equivalent of infinite masses, the local oscillator is said to have "sharp" natural frequencies. If the scatterer mass is reduced from infinity to a finite value (as illustrated by the vertical dashed line in Fig. 6) then a mass might be moved by the motion of one local oscillator string, and affect the next local oscillator. In other words, the case of finite scatterer mass means that the local oscillators become coupled together. The result is analogous to what happens when two or more simple mass-spring oscillators are coupled together. A number of new normal modes and natural frequencies arise. Thus the result of coupling the local oscillators is that the sharp frequencies of an isolated oscillator broaden out into bands, and band structure appears (as seen by moving from the far right toward the left in Fig. 6). In solid state physics, this picture of the evolution of frequencies from the sharp values of isolated local oscillators into bands is called the "tight binding" model.[4] In an isolated atom, an electron is tightly bound to an ion core, and because of quantum mechanics, the electron has discrete (sharp) energy levels. But when many atoms are put together at periodic lattice sites in a crystal, the electrons may interact with neighboring atoms, and the sharp atomic energy levels broaden out into bands, resulting in the electronic band structure in solid state physics. It should be noted that nearly all modern electronic devices owe their existence to the band structure of semiconductors.

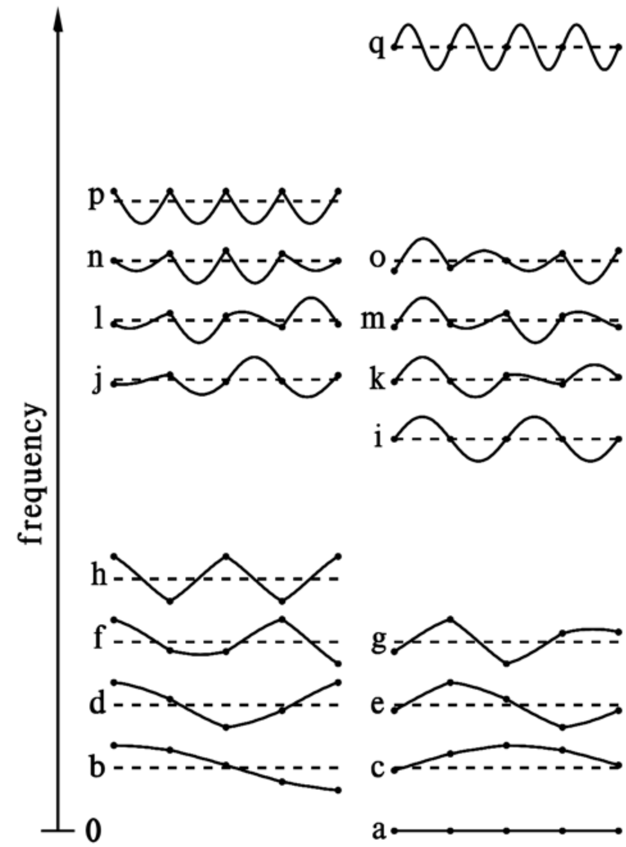

Fig. 7. The modes of vibration for a string with scatterers of finite mass. The vertical position of the dashed line for the different modes is roughly proportional to the frequency of the mode. Some modes remain degenerate. The gaps between modes "h" and "i" and between modes "p" and "q" are apparent.

Mode shapes and wave propagation

As discussed above, the symbols along the vertical dashed line in Fig. 6 give the frequencies for the modes of a string that contains periodic scatterers of finite mass *m*. What about the nature of the modes themselves? When the scatterer mass was zero, the modes could readily be drawn as in Fig. 5a. When the scatterer mass was infinite, the modes could readily be drawn as in Fig. 5b. However, when the scatterer mass is finite, determining the modes is not at all easy. To draw the modes for the string with finite masses, equivalent to the modes "a" through "q" in Figs. 5 and 6, a computer program was used with the results shown in Fig. 7. The vertical position of the dashed line for the different modes in the figure is roughly proportional to the frequency of the modes. The gaps between modes "h" and "i" and between modes "p" and "q" are apparent. Some modes, such as pairs "b" and "c," "d" and "e," etc., remain degenerate. However, for the plain string the pair of modes "h" and "i" and the pair "p" and "q" were degenerate, but now the presence of finite scatterers has "split" these degeneracies, resulting in the gaps.

For a system with finite mass scatterers, there is an important type of diagram, called a band structure diagram (Fig. 8) that is used to display graphically some of the information indicated by the symbols along the vertical dashed line in Fig. 6 and by the features of Fig. 7. More information concerning the band structure diagram, mode modulus, dispersion, and Bloch waves may be found in the Appendix.

An important feature of wave propagation in periodic media is that there can be no wave propagation at any frequency that falls in a stop band or gap. For example, there can be no propagating wave at a frequency that lies between "h" and "i" and between "p" and "q" in Fig. 7. However, when a wave at a gap frequency, traveling in a region free of scatterers, encounters a periodic array of scatterers, the wave motion is not stopped abruptly. The wave enters the region of scatterers as an "evanescent" wave[7] that decays exponentially. Because there is no energy transmission for an evanescent wave, the energy of the incident wave is completely reflected.

New research with waves in arrays of scatterers

Interest in wave propagation in periodic systems dates back to the 1880's. Lord Rayleigh published an article on waves in periodically stratified media in 1887.[8] Later, the subject intensified considerably with the application of coherent electromagnetic waves (microwaves and light from lasers) in systems with periodic elements. By the 1970's virtually all aspects of waves in periodic systems were understood and applied in many clever devices. An article reviewing the subject, written in 1976 by C. Elachi,[9] contains nearly 300 references. With the advent of semiconductor devices in the 1960's, accurate methods for calculating the band structure of complicated three-dimensional periodic systems were developed.[4] Currently, there is little that could be added to the understanding of wave propagation in periodic systems.

New possibilities arise, however, if one studies systems that deviate from the confines of static, linear and exactly periodic systems. For example:

(1) No real system can be perfectly periodic, so it is important to understand wave propagation in arrays of scatterers that are disordered. The disorder may range from slight, where random deviations from perfect periodicity are small, to significant, where disorder is characterized by a broad distribution function. The appendix shows how the modulus of the modes of Fig. 7 may be derived. The modulus is illustrated in Fig. 9 and is the same in every unit cell throughout an infinite system. If a system deviates randomly from perfect periodicity, then one might expect that the new wavefunctions would show random variations in the moduli in different unit cells. However, this is not what happens. Instead, the wavefunctions become exponentially localized; that is, the modulus of the wavefunction has a maximum value at one unit cell, and then the modulus decays exponentially with distance from that site. This quite unexpected behavior is called Anderson localization, and it was cited in the 1977 Nobel Prize of Philip Anderson and Sir Neville Mott.[10] Thus wave propagation in disordered arrays of scatterers is a significant matter indeed.

(2) There are configurations of scatterers, referred to as quasicrystalline,[3] that are intermediate between periodic and disordered. For one-dimensional disordered or quasicrystalline configurations, there are rigorous theorems, analogous to Floquet's theorem, that govern the behavior of wave propagation in such systems. However, in two and three dimensions, no one has been able to prove any theorems, so the behavior of waves in higher dimensional disordered or quasicrystalline systems has been a viable field of research.[2,3]

(3) In the article so far we have assumed that the positions of the scatterers (periodic or disordered) were static, i.e., they did not change in time. However, there are impor-

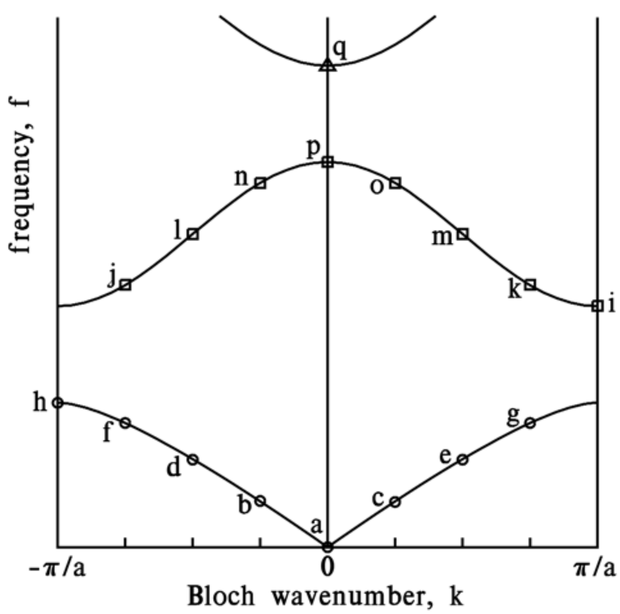

Fig. 8. Band structure diagram for the example with l/a = 4. The letters refer to the modes in Fig. 7. The frequencies and their symbols are taken from the modes in Fig. 6.

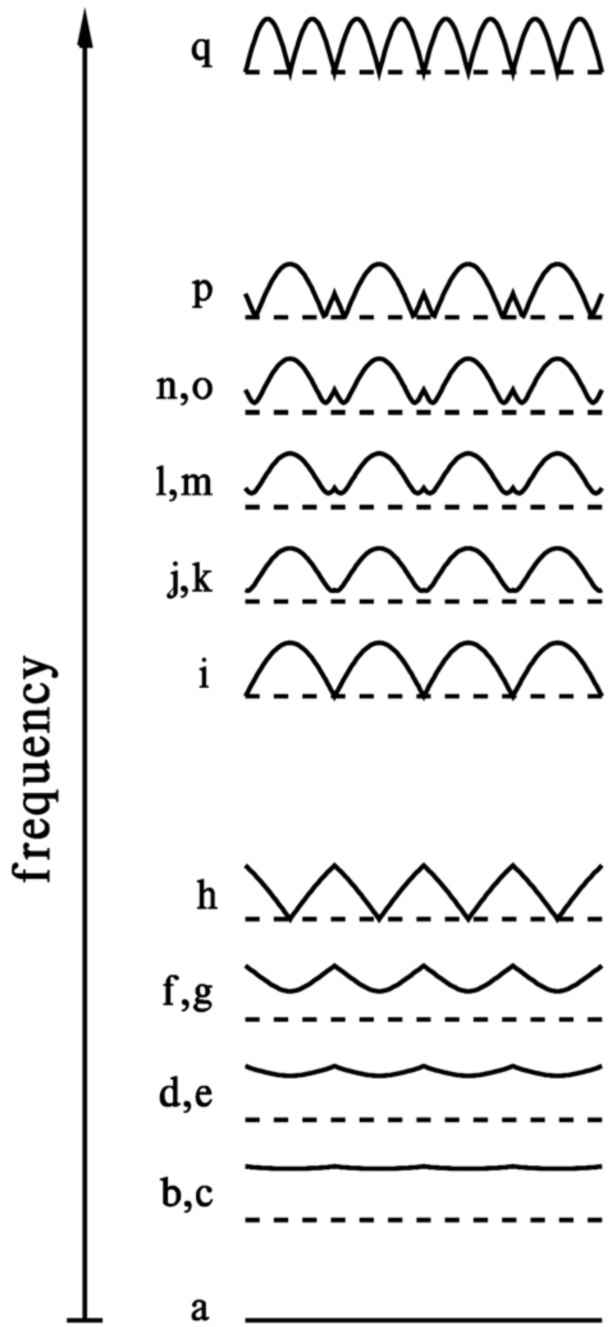

Fig. 9. Plots of the modulus of the modes from Fig. 7. Taking the modulus results in just the periodic part of the mode, $|U_{k,n}(x)|$, with the period being the lattice constant (one-fourth of the length of string shown in the plot). Double letters refer to degenerate modes which were combined to give the modulus.

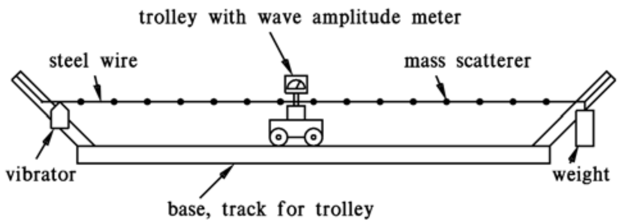

Fig. 10. Apparatus for demonstrating wave propagation in periodic or disordered arrays of scatterers. The wave medium is a steel wire (guitar string), with tension maintained by a weight, and with scatterers consisting of small masses (split shot for fishing line) positioned along the wire.

tant systems of scatterers where the positions are time-dependent. An important example occurs when the scatterers are the atomic ions in a metallic crystal and the waves are the quantum mechanical Schrödinger waves of the electrons. In this case the positions of the scatterers are changing because of the thermal motion of the ions. Another example would be when the scatterers are small particles suspended in a fluid. Here, the scatterers change positions because of another thermal effect—Brownian motion.

(4) Although not explicitly stated, all of the results discussed so far have been obtained assuming that the wave propagation in the system is linear; that is, none of the equations governing the system contain terms that are not linearly proportional to the wavefunction. However, if a wave system is driven to sufficiently high amplitude, then classical force laws will result in the appearance of nonlinear terms. For example, the wave equation for a string is based on the assumption that the tension in the string is constant. However, if the string has a finite transverse displacement, then the arc length of the string is increased, and if the string is rigidly clamped at two ends, then the presence of the finite amplitude wave will increase the tension. This will cause a nonlinear term to appear in the wave equation.

Each of the deviations discussed above have individually been the subject of significant research. However, more possibilities arise when one studies systems that involve combinations of these deviations, and for these systems theoretical treatment, and even numerical treatment, is very difficult. In these cases, there is significant opportunity for very interesting experimental research. An important consideration is that if such experimental research is to be undertaken, then the experiment must be able to pass a crucial test—the experiment should be first configured with a static and periodic (as accurately as possible) array of scatterers and a linear wave medium, and under these conditions the experiment must be able to verify Floquet's theorem. Only when this test is passed may the experiment be considered valid for the study of deviations. Thus, the understanding of wave propagation in periodic systems maintains its importance even in novel experiments.

An acoustical experiment

An experimental apparatus that is used to demonstrate phenomena in both periodic and disordered arrays of scatterers is illustrated in Fig. 10; it is a realization of the model discussed in this article, involving a string with masses as scatterers. A larger version, with more than 50 masses, was used in a research program to study the additional effects of time-varying mass positions[11] and finite amplitude (nonlinear) transverse waves.[12]

As shown in Fig. 10, the experimental apparatus has a base with struts angled at 45 degrees at each end. A steel wire (a guitar string) is attached between the extreme ends of the struts. Starting at the left end in the figure, the wire travels at 45 degrees until it passes through an eyelet and proceeds horizontally; an electromechanical vibrator is attached at this point. The wire continues horizontally to the right end where a large weight is attached, after which the wire continues at 45 degrees to the attachment point on the right strut. The large weight is supported by the wire only, and given the configuration of the wire as shown in the figure, the weight maintains constant tension in the wire. Positioned along the horizontal section of the wire are small masses ("split shot" for fishing line) whose positions are carefully set using precision calipers.

The base of the apparatus acts as a track for a trolley that carries a wave amplitude detector and meter, fashioned from an electric guitar pick-up and a modified sound-level meter. As the trolley is moved along the wire, the meter indicates the transverse displacement of the wire as a function of the horizontal position, i.e., $\psi(x)$. A photograph of the detector, with the wire and some masses just visible, is presented in Fig. 11.

A model of a second apparatus that could be used to study two-dimensional phenomena is shown in Fig. 12. The local oscillators in this apparatus are musical tuning forks

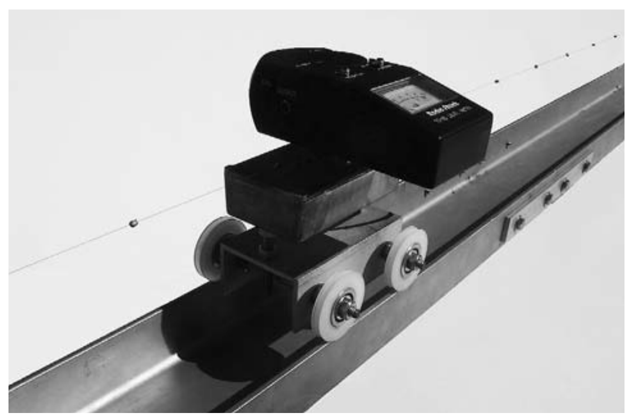

Fig. 11. Photograph of part of the apparatus shown schematically in Fig. 10.

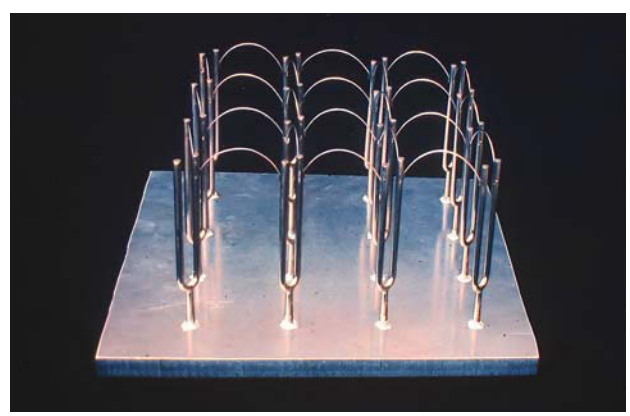

Fig. 12. An apparatus used to demonstrate phenomena for wave propagation in two-dimensional arrays of scatterers. The model is based on musical tuning forks as local oscillators, which are coupled together with arcs of steel wires attached to the tines of the tuning forks.

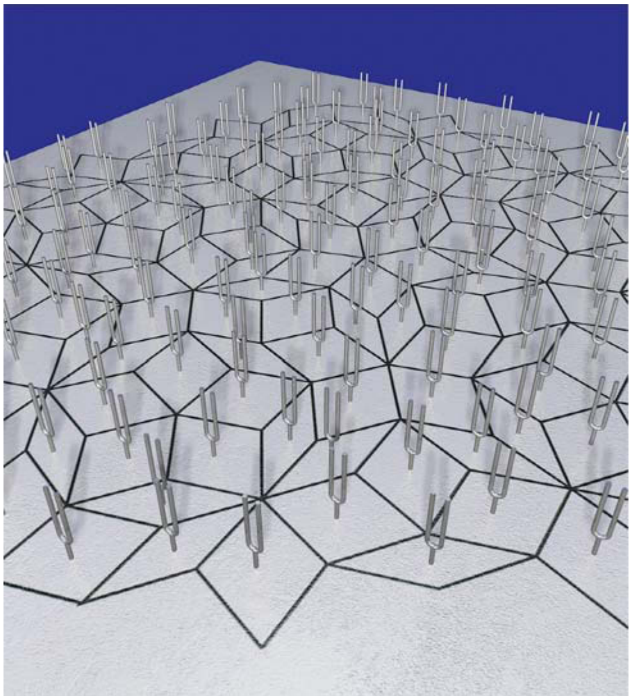

Fig. 13. A graphic rendition of the tuning-fork, Penrose-tile, two-dimensional quasicrystal. This particular Penrose-tile is formed with a pattern of two unit cells—a skinny rhombus and a fat rhombus.

that all have a "sharp" frequency of 440 Hz. The local oscillators are coupled together with arcs of steel wire spot-welded at the ends of the tines of the tuning forks, as shown. The coupling causes the sharp frequency to broaden out into a band. While the system is fairly complicated, the periodic (square lattice) nature and the effects of the periodicity on the natural frequencies for the system are quite definite. This system was used to see if the array of tuning forks could correctly produce the predictions of Floquet's theorem.

The square lattice of tuning forks pictured in Fig. 12 was actually a test case for extending the research to a two-dimensional quasicrystalline system. In this case, the square periodic lattice was replaced with a quasicrystalline lattice known as a "Penrose tile."[13] One way of describing a periodic system is to specify its construction as "taking a unit cell and repeating it to fill space." To construct a Penrose tile, one is allowed to use more than one unit cell to fill space.

A graphical rendition of an actual tuning fork Penrose tile is shown in Fig. 13. As may be observed, this particular Penrose tile is formed with a pattern of two unit cells—a skinny rhombus and a fat rhombus. The tuning fork array experiment was used to provide the first evidence of the effects on wave propagation in a two-dimensional Penrose tile quasicrystal.[14]

The author is planning to prepare additional tutorial articles on wave propagation in disordered and quasicrystalline arrays of scatterers.

Appendix

To understand a band structure diagram, it is helpful to examine the mathematical form of the modes $\psi(x)$ of a periodic system. From Floquet's theorem,[5] it is found that the function $\psi(x)$, with some subscripts added, must be of the form

$$\psi_{k,n}(x) = e^{ikx}U_{k,n}(x) - \frac{k}{|k|}e^{-ikx}U_{-k,n}(x) \quad (1)$$

where k is an important parameter called the Bloch wavenumber. In Eq. 1, n is an integer indexing different bands (e.g., in Fig. 6, the circles have $n = 1$, the squares are for $n = 2$, etc.), and $U_{k,n}(x)$ is a function that is periodic in x with the period equal to the lattice constant a, so that $U_{k,n}(x+a)=U_{k,n}(x)$. Solutions as in Eq. 1 are called Bloch wavefunctions.

The Bloch wavenumber must lie within the range $-\pi/a$ to π/a, and it must have discrete values given by positive or negative integer multiples of π/l. With $l/a = N$, k may have $2N+1$ possible values. However, $k = -\pi/a$ and $k = +\pi/a$ generate exactly the same mode, so that there are effectively only $2N$ values for k. In the example illustrated in Figs. 6 and 7, $l/a = 4$, so that k may have eight possible values. It should now be noted from Fig. 6 that each band has eight distinct modes: "a" through "h" for the first ($n = 1$) band, and "i" through "p" for the second ($n = 2$) band. Thus the Bloch wavenumber is an index to different modes within each band. Now the band structure diagram can be described—it is a plot with the Bloch wavenumber indexing the possible modes on the horizontal axis (with $2N$ discrete values between $-\pi/a$ to π/a), and the frequencies of the modes, one band after another, on the vertical axis. For the example with $N = l/a = 4$, the band structure diagram is shown in Fig. 8. The letters refer to the modes in Fig. 7, and the frequencies and their symbols are taken from Fig. 6.

It should be noted that in Fig. 8, the symbols for modes "h" and "i" could be placed on either side of the plot, because $-\pi/a$ and π/a are equivalent. These modes were plotted where they are in Fig. 8 simply to maintain the back-and-forth pattern of the modes in the figure. It should also be noted that if $N = l/a$ is increased, then the curves in Fig. 8 remain the same, but the number of modes in each band increases, and the density of symbols on the curves for each band increases. For a "large" system ($l \rightarrow \infty$), the band structure is given by the continuous curves in Fig. 8; in this case the vertical extremes of the continuous curves are referred to as "band edges."

Another important feature of wave propagation in periodic media is that there is non-trivial dispersion; that is, the frequency is a nonlinear function of the Bloch wavenumber. For the plain string with no scatterers, the Bloch wavenumber becomes $k=2\pi/\lambda$, and the dispersion is linear with $f = ck/2\pi$. For a string with scatterers, the frequency as a function of Bloch wavenumber is given by the nonlinear curves of Fig. 8. The non-trivial dispersion of waves in periodic systems may be used to advantage in many devices.[9]

The limits of the Bloch wavevector at $-\pi/a$ and π/a is referred to as the "Brillouin zone boundary."[1,4] In two and three dimensions, the nature of band structure and Brillouin zone boundaries is very interesting; a systematic approach to periodic wave systems in higher dimensions was one of the major contributions of Leon Brillouin.[1]

Another interesting plot that may be made involves the modulus of the modes. For standing wave modes that are not degenerate, the modulus is simply given by the absolute value of the mode: $|\psi_{k,n}(x)|$. For standing wave modes that are degenerate, then the modulus is given by the square-root of the sum of the squares of the two degenerate modes. It can be shown that the modulus of a mode gives a result that is proportional to the periodic part of the Bloch wavefunction: $|U_{k,n}(x)|$. The results for the moduli of the modes in Fig. 7 are shown in Fig. 9. The periodic nature of the modulus of the Bloch wavefunctions is apparent in the figure, and one may note a systematic pattern as the modes are followed upward, as the frequency is increased. Because the function $|U_{k,n}(x)|$ is periodic, the modulus of the wavefunction has the same maximum value in every unit cell throughout the entire system; for this reason, the wavefunctions of a periodic system are said to be "extended."

The presentation given above has used standing Bloch wavefunctions as graphical examples. However, traveling Bloch wavefunctions are more commonly used in other treatments—these are given by

$$\psi_{k,n}(x) = e^{ikx} U_{k,n}(x) \qquad (2)$$

A traveling Bloch wavefunction moving in the opposite direction is obtained by replacing k with –k. It can be shown that this is equivalent to taking the complex conjugate of Eq. 2. From Eq. 1 it can be seen that the Bloch standing wavefunction is a linear combination of two Bloch traveling waves going in opposite directions, just as for waves on a plain string. However, it is important to note that the presence of the factor $\exp(ikx)$ or $\exp(-ikx)$ in a Bloch wavefunction does not mean that the motion of the system is that of a traveling wave with wavenumber k. The functions $U_{k,n}(x)$ are complex and also involve a phase, so that the actual motion of the system is complicated. If the scatterers are strong (large m), then the actual motion of a Bloch wave is nearly that of standing waves between the scatterers, and the factor $\exp(\pm ikx)$ simply represents a phase change in proceeding from one local standing wave to another. Ordinary traveling waves have certain relations with real force, velocity, momentum, energy, etc., and Bloch waves do not have straightforward relations with such quantities. For example, a Bloch wave traveling in one direction may have energy traveling in the opposite direction. The same caution in interpreting $\exp(ikx)$ may be found in textbooks on solid state physics, where it is noted that Plank's constant times the Bloch wavenumber k is not real momentum, but is instead something referred to as "crystal momentum."[4]

While Bloch waves may not be traveling waves in a system, it is possible to form wave packets that do behave like ordinary wave packets. As shown in modern physics texts,[15] wave packets travel at the group velocity given by $c_g = 2\pi df/dk$. Most acoustic systems have linear dispersion, with $f=ck/2\pi$, so that $c_g = c$. However, as discussed earlier, Bloch waves have non-trivial dispersion given by the curves in Fig. 8. In the second band (and every other band) the group velocity is negative, so that wave fronts in a wave packet may move in one direction while the wave packet itself moves in the opposite direction.

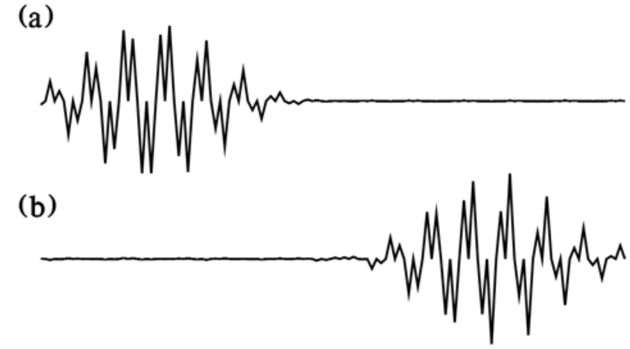

Fig. 14. Illustration of the propagation of a Bloch wave packet. The two plots (a) and (b) illustrate the propagation of the packet from one instant in time to a later instant in time. In the context of the tight binding model, it is interesting to see that the local oscillators give up their energy in just the right manner.

Given the tight-binding picture of waves in periodic systems, it is surprising that it is possible to have a traveling wave packet at all. With strong scatterers, the acoustic energy is mostly stored in a standing wave for a local oscillator, with only the phase $\exp(ikx)$ changing from one local oscillator to the next. To have a traveling wave packet, a set of local oscillators must give up all of their energy to the next set of local oscillators in the direction of propagation. A computer calculation showing that this actually works is shown in Fig. 14. In this figure, the upper plot is the Bloch wave packet at one instant of time, and the lower plot is the packet at a later instant in time. The periodic scatterers are not shown in this plot, but there are about 64 unit cells across the figure.**AT**

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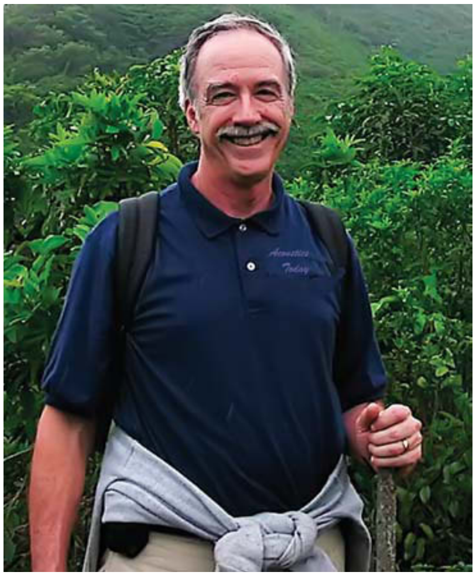

Julian D. Maynard, Jr. received his undergraduate degree at the University of Virginia in 1967 and his Ph.D. in physics at Princeton University in 1974. Jay joined the faculty at The Pennsylvania State University in 1977 and currently has the title of Distinguished Professor of Physics. His acoustics research has been featured in the *New York Times* Science Section, and has appeared in *Physics Today, Reviews of Modern Physics, New Scientist Magazine, La Recherche* and *Physik in unserer Zeit.* A film about his invention, Nearfield Acoustical Holography, developed with E. G. Williams, has been aired several times on the PBS television series *Nova*. His research is referenced in the textbook *Superfluidity and Superconductivity* and was cited in the *50 Years of Physics in America* issue of *Physics Today*. Professor Maynard is a Woodrow Wilson Fellow, Alfred P. Sloan Fellow, Fellow of the Acoustical Society of America and the American Physical Society, and a recipient of the Silver Medal in Physical Acoustics from the Acoustical Society of America.