WAVE PROPAGATION IN ARRAYS OF SCATTERERS TUTORIAL: PART 2

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This is the second part of a tutorial series on wave propagation in arrays of scatterers. The first part[1] of the tutorial discussed wave propagation in arrays of scatterers arranged periodically; this part will cover waves propagating in disordered (non-periodic) arrays. A later part will treat waves propagating in a relatively new type of array: the "Penrose tile" or "quasicrystal."[2,3]

"The problem of wave propagation in disordered systems remains open, but some very important general results have been established."

A periodic array of scatterers is relatively easy to treat, because the periodic array is formed with a "unit cell" repeated to fill space, and it is only necessary to solve a scattering problem for one unit cell. The results are based on rigorous theorems: Floquet's theorem[4] for one dimension and Bloch's theorem[5] for two and three dimensions. Wave propagation in a disordered array is considerably more difficult, because now it is necessary to solve the full problem, satisfying boundary conditions on each of a large number of scatterers. The problem dates back to the era of Lord Rayleigh—1869–1919—(with interest in sound propagation through forests, etc.), and since that era a significant number of distinguished scientists have worked on the problem. There was some progress, but nothing like a complete understanding.

More recent interest in waves in disordered systems arose in condensed matter physics, with the development of sub-micron sized components for integrated circuits for computers, etc. The connecting wires in integrated circuits are non-crystalline metals, and within metals electrons move as waves (as prescribed by quantum mechanics) scattered by the disordered array of ions. Ordinarily, thermal motion of the ions would wash out the wave character of the electrons, so the problem could be treated relatively easily and accurately with a diffusion equation. However, if a wire were sufficiently small, then an electron could maintain its wave nature, and it would then be necessary to understand wave propagation in a disordered array of scatterers. This area of condensed matter research, referred to as "mesoscopic physics" or "long-range phase coherence" led to significant advances in understanding waves in disordered systems.[6]

The first rigorous theorem (analogous to Floquet's theorem for periodic arrays) applicable to wave propagation in a disordered array was found by the mathematician Hillel Furstenberg in 1963.[7] Furstenberg's theorem is relatively recent on a time scale which dates back to Lord Rayleigh, so wave propagation in disordered arrays has been a long and difficult problem. But Furstenberg's theorem treats only one-dimensional systems, and unlike Floquet's theorem, it cannot be extended into two or three dimensions. So the problem of waves in disordered systems remains open. Despite the difficulty in high dimensions, some very important general results have been established. These results, together with the solved problem in one dimension, are the subjects of this article.

One-dimensional disordered systems

The starting point for the discussion of waves in disordered systems will, of course, be the solved one-dimensional case. An essential first step will be to review the results for waves in one-dimensional periodic systems, as presented in the first part of this tutorial series. Some of those results may be summarized[1] as follows:

1. The model used for illustrating the results is shown in Fig. 1a; it is a string, with some mass per unit length, stretched to a constant tension. The waves are transverse waves which travel with the characteristic speed for the string. The scatterers are point masses separated with a lattice constant, *a*. One mass and one section of string, of length *a*, comprise a unit cell.

2. The wave field solutions (which may be referred to as eigenfunctions, normal modes, wave functions, etc.) for periodic systems are "extended;" that is, the modulus of the eigenfunction is the same in every unit cell. This is illustrated in Fig. 1b.

3. A simple "balls-and-springs" understanding of the extended eigenfunctions is that each section of string between pairs of masses acts as a "local oscillator," and for the periodic system all of the local oscillators are identical and have identical "local oscillator frequencies." When the system is excited near a local oscillator frequency, any energy (how-

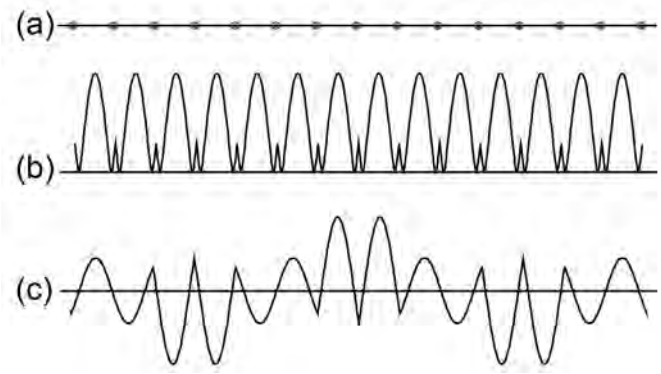

Fig. 1. Review of the model system and results for wave propagation in a one-dimensional periodic array of scatterers. (a) Illustration of the model system, consisting of a string with masses (scatterers) spaced periodically. (b) The modulus of the wave field solution for a periodic system; since the modulus is the same in every unit cell, the wave field is said to be "extended." (c) The wave field itself is more complicated due to interference effects.

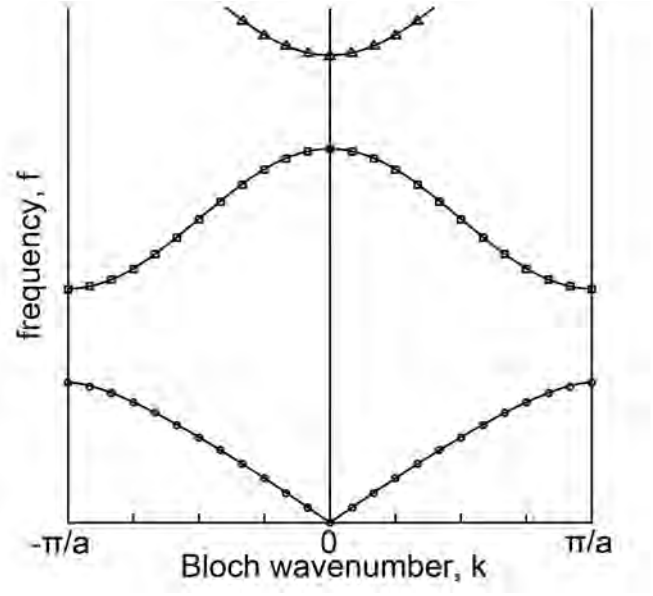

Fig. 2. A band structure diagram providing eigenfrequencies as functions of Bloch wavenumber and band index (represented by the different symbols).

ever small) which may be transmitted past a scatterer will excite the next local oscillator into resonance, and that oscillator can grow to large amplitude. This is the case for all of the local oscillators, and hence the extended eigenfunction is possible.

4. Unlike the modulus, the eigenfunction itself (the real or imaginary part of a complex field) can exhibit interference phenomena, and have a non-trivial appearance, as illustrated in Fig. 1c.

5. Every eigenfunction has an associated eigenfrequency (or natural frequency). Each eigenfunction and eigenfrequency can be labeled with two parameters: a "Bloch wavenumber" and a "band index." The eigenfrequencies may be displayed graphically in a "band structure" diagram, as illustrated in Fig. 2. Discrete values along the horizontal axis are the Bloch wavenumbers, and the lines of connected symbols in Fig. 2 are the bands; counting the individual bands from the bottom upward gives the band index. The bare horizontal regions between the bands are called "gaps" or "stop bands."

6. For a periodic system with N local oscillators, each band will have N discrete eigenfrequencies.

7. Waves may propagate at frequencies within a band, but not at frequencies which fall in a gap. If a wave at a gap frequency is incident on an interface between a uniform medium and a periodic array, then the wave cannot propagate into the array; however, the field does not stop abruptly, but can extend into the array as an evanescent wave[8] decaying exponentially into the array region.

With the features of eigenfunctions in a periodic array reviewed in items 1 to 7 above, we can now consider what happens with a disordered array of scatterers. The model for a one-dimensional disordered system will be similar to the periodic system in Fig. 1a, but instead of the scatterers having a constant lattice spacing, a, the spacing between the scatterers varies (i.e., the spacing is random), as illustrated in Fig. 3a. The system may still be thought of as consisting of coupled local oscillators, but now each local oscillator is different. With reference to item 3 above, instead of having identical local oscillators giving the same amplitude at each site, it would be expected that different local oscillators would give different amplitudes at each site. However, this is not at all what happens—what happens is that there is a maximum amplitude at one site, and moving away from that site, the amplitude of the eigenfunction exponentially decays, as illustrated in Fig. 3b. The eigenfunctions for a disordered array of scatterers are "exponentially localized."

The occurrence of exponential localization may seem quite unexpected—why does disorder in the local oscillators not simply duplicate Fig. 1b but vary the amplitude at different locations? The fact that the eigenfunctions become exponentially localized is the essence of Furstenberg's theorem. It is a very significant result, and it was a crucial element in the 1977 Nobel Prize for Philip Anderson and Sir Neville Mott. The occurrence of exponentially localized eigenfunctions in one-, two- and three-dimensional disordered systems is now referred to as "Anderson localization." This phenomenon and other advances in solving very difficult problems in long range phase coherence in disordered systems are very important results, and they have been the basis of landmark papers and numerous awards in condensed matter physics.

What is desired is a "balls-and-springs" explanation of how disorder leads to exponential localization of eigenfunctions. In violation of the experimentalist's creed that "six months in the lab can save you a day in the library," it is possible to go to the library and search for papers on Anderson localization. There are very many such papers, but nearly all use Anderson localization as a way of explaining some phenomenon in a disordered system, but none provide the explanation for exponential localization. A look at Furstenberg's paper shows that a researcher must be well versed in the preceding developments in that field of mathematics to gain any insight. The consequence of Furstenberg's theorem for waves in a one-dimensional disordered system can be stated simply—proceeding to plus and minus infinity, eigenfunctions of a disordered system decay exponentially to zero with "probability one." But understanding the manifestation of that statement and its probabilistic nature in a real experimental system of finite size is not easy—there is no "balls-and-springs" explanation.

While there is no simple physical explanation, it is possi-

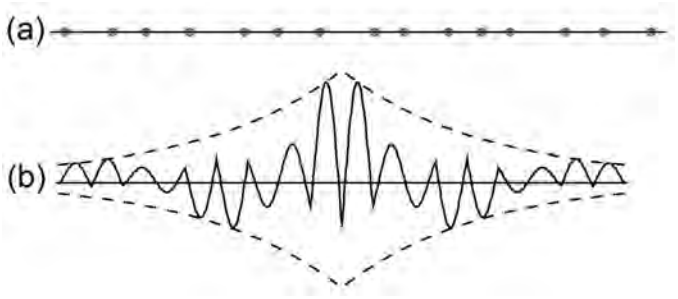

Fig. 3. The model for a disordered array of scatterers and an exponentially localized eigenfunction. Unlike Fig. 1a, the scatterers in (a) do not have the same lattice spacing, so the local oscillators are different. The question is: how does the disorder lead to the exponential localization, as illustrated in (b)?

ble to gain significant understanding by seeing how to calculate a localized eigenfunction (and its eigenvalue) in an actual one-dimensional disordered system. While it may not be necessary except for a deeper understanding, it will be worthwhile to review the mathematics which underlies the method for finding such an eigenfunction. Insight into Anderson localization can be gained even if the mathematical notions simply establish terminology. In any case, we will "bite the bullet" and delve into mathematics; the useful concepts are as follows:

1. The one-dimensional wave equation is a linear differential equation involving second order derivatives. The differential equation will involve a parameter such that when boundary conditions are added (from the array of scatterers), then solutions may be found only for certain values of the parameter; this parameter becomes the eigenvalue, and the solution for that value is the eigenfunction $\psi(x)$. For such a problem there are many math equations comprising what is known as Strum-Liouville theory.[9]

2. It will be helpful to use the concept of "linearly independent functions"—two functions $f_1(x)$ and $f_2(x)$ are linearly independent if and only if it is impossible to find two nonzero constants C_1 and C_2 such that $C_1 f_1(x) + C_2 f_2(x) = 0$ for all x. Despite the mathematical formality of the definition, such functions are common in acoustics; examples of pairs of linearly independent functions are $\sin(kx)$ and $\cos(kx)$, $\exp(ikx)$ and $\exp(-ikx)$, and $\exp(\alpha x)$ and $\exp(-\alpha x)$. It should be noted that the last pair may represent evanescent waves.[8]

3. For the wave equation there exists two linearly independent solutions, and any solution may be expressed as a linear combination of the two.

4. If one linearly independent function exponentially decays asymptotically, then the second linearly independent function will exponentially grow asymptotically. The important notion here is that for sufficiently extreme values of some variable, one of the two linearly independent solutions will be much larger than the other.

5. If the value of an eigenfunction and its first derivative are known at just one point [$\psi(x_0)$ and $d\psi/dx\,(x_0)$ for some x_0], then that is sufficient information to determine the eigenfunction $\psi(x)$ at all points.

6. Furstenberg's statement about exponential localization (with probability notions in an infinite system) will be assumed, so that we have some idea of the properties of the eigenfunction. It should be noted that the exponential localization does not mean that the eigenfunction is given asymptotically by $A \exp(-\alpha |x|)$, but rather that its amplitude is bounded by that function, for some value of A.

We now have enough mathematical concepts (or at least terminology) to develop an understanding of Anderson localization in one dimension. We begin by supposing that we know the exact value of one eigenvalue as well as the exact value and first derivative of its eigenfunction at one point. Then by statement 5 above, we can find the entire eigenfunction. This is illustrated in Fig. 4a—the point where the value and first derivative of the eigenfunction are known is in the center. The math theorem means that the entire wave function can be found by proceeding in the positive and negative x-

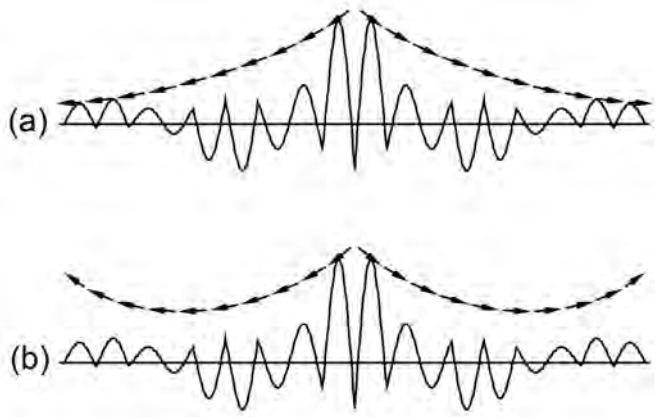

Fig. 4. Illustration of the method for determining a localized eigenfunction from the value and first derivative of the eigenfunction at a point (at the center in the illustrations). (a) Using exact values and math theorems. (b) Using a numerical method in a computer.

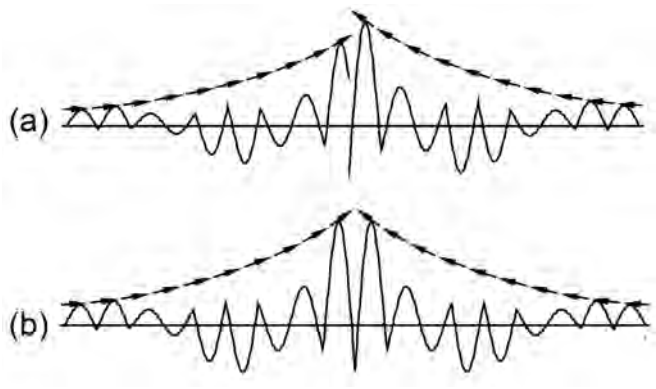

Fig. 5. Illustration of the numerical method for finding eigenvalues and eigenfunctions using a computer. The method iterates inward and finds solutions which grow into dominance, although the iterations may not match in the middle, as in (a). The eigenvalue is found by requiring the iterations to match their value and first derivative upon crossing, as in (b).

directions, as suggested by the arrows in Fig. 4a. It should be noted that having the exact starting value and first derivative of the eigenfunction causes coupling to just the linearly independent solutions which have the required exponential decay.

Now consider using a computer to determine the localized eigenfunction numerically. Starting at the center, a reliable computer algorithm[10] may be used to proceed in the positive and negative x-directions to compute the eigenfunction iteratively. However, the exact starting value and first derivative of the eigenfunction must now be represented with a number in the computer having a finite number of digits. This means that the starting condition, while mostly coupling to the linearly independent solutions which exponentially decay, must now contain some small admixture of the linearly independent solutions which exponentially grow. The consequence is that in moving away from the starting point, the linearly independent solution which exponentially grows, despite having a small start, will eventually dominate the calculation and cause significant deviation from the desired eigenfunction, as illustrated by the upward curvature of the arrows in Fig. 4b. How to avoid this problem, and more significantly, how to find the eigenfunction without knowing the eigenvalue or the value and first derivative at a point, will be presented next.

Ironically, the method for finding the eigenfunction (and its eigenvalue) is to turn the exponential dominance of one linearly independent solution over the other into an advantage. First, select an arbitrary value for the eigenvalue parameter in the differential equation. Next, begin with a large value of the argument x and arbitrary numbers for the starting value and first derivative at this point. Then iterate in the direction of decreasing x until the growing linearly independent solution dominates. This process is illustrated by the arrows pointing left in Fig. 5a. At the same time, start with a small value of x and iterate in the direction of increasing x until the other linearly independent solution dominates, as indicated by the arrows pointing right in Fig. 5a. At the point where the left and right iterations cross, the true eigenfunction should have a matching value and first derivative. Since our choice for the eigenvalue parameter in the differential equation was just a guess, the matching of the iterations is unlikely, as illustrated in Fig. 5a. However, we can change the eigenvalue guess and try again. Indeed, we can utilize a computer search routine to efficiently find the eigenvalue which causes the iterations to match, with the matched iterations comprising our solution for the eigenfunction; such a success is illustrated in Fig. 5b.

The above process may be repeated to find other eigenvalues and eigenfunctions. It is important to note that by iterating in the inward direction and finding the linearly independent solutions which grow, we fortuitously find the localized eigenfunction which exponentially decays in the outward directions, toward plus and minus infinity.

We can now address the notion of "probability one" at "plus and minus infinity" in Furstenberg's theorem. In the reverse iterative process for finding the localized eigenfunction, the probabilistic nature is manifest in the assumption that the iterations are taken far enough so that one of the linearly independent solutions is probably much larger than the other one. Furthermore, although we needed sufficiently large and small starting positions x for the arguments in the iteration, the system for which we have found exponentially localized eigenfunctions is finite in size. Thus this method for actually finding exponentially decaying eigenfunctions gives some insight into the nature of Anderson localization in one dimension.

Anderson localization in two and three dimensions—The scaling argument

The properties and consequences of Anderson localization in one dimension are firmly established with rigorous theorems. As already mentioned, in two and three dimensions there are no analogous rigorous theorems. There is a "scaling argument"[11,12] which treats one, two, and three dimensions, and this argument provides an adequate explanation for the behavior of electrons in disordered solids. However, the scaling argument is based on an assumption which may be valid for electrons in solids at finite temperatures, but which may be less convincing for ideal (lossless) waves in disordered systems in the regime of Anderson localization. Some of the results of the scaling argument are known to be incorrect in one dimension,[13] and one of the scaling argument results in two dimensions has been ques-

tioned. There have since been large scale computer simulations which reveal that, while the predictions of the scaling argument may be correct, the phenomenon of percolation may also play a role.[14]

The scaling argument uses renormalization group[15] techniques to examine how the conductivity of a finite-sized sample of the material varies as the sample size is systematically scaled up toward infinite size. A key assumption of the argument is that as the size of the sample becomes infinite, inelastic scattering due to finite temperatures destroys the wave nature of the electrons so that the diffusion approximation becomes valid. However, the fundamental problem of waves in a static disordered system, where there is absolutely no inelastic scattering or attenuation of the waves, even at boundaries, remains a challenging problem. In any case, the scaling argument is important for systems which have some inelastic scattering. One of the most important results is that as the amount of disorder in a three-dimensional system is increased, there is a transition from a situation where at least some eigenfunctions are extended, to a situation where all the eigenfunctions are localized; this is called the Anderson transition. Two dimensions is referred to as the critical dimension, where the theory is particularly difficult; in a two-dimensional system with some disorder, it is believed that all eigenfunctions are localized (as for one dimension), so that there is no Anderson transition.

Coherent backscatter

When a plane wave in a uniform medium is incident upon the surface of a disordered system, then one might expect uniform diffuse reflection. But again the disordered system presents a surprising result. In this case it is found that the waves experience enhanced reflection in a direction opposite to the incident direction, and this is called "coherent backscatter."[16,17]

Coherent backscatter was first noted in regard to the brightness of the lunar surface at full moon, when the observed reflection is nearly opposite the illumination. Lunar as well as laboratory observations have been made since 1922, and the effect was theoretically explained as due to reflection from a disordered medium by Hapke[18] in 1963. Experiments designed to specifically test Hapke's theory were made by Oetking[19] in 1966. More extensive measurements and a re-working of the theory were published by Ishimaru, *et al.* in 1984.[20,21]

Coherent backscatter was rediscovered by a number of researchers around 1985.[22–24] This time the theoreticians involved were also engaged in studies of Anderson localization, and the effect was referred to as "weak localization" and was made very popular. The term "weak localization" is misleading because the effect occurs regardless of whether or not eigenfunctions are localized. As will be seen below, coherent backscatter is a direct consequence of time-reversal invariance, whereas, as Furstenberg's theorem illustrates even for one-dimension, localization is a much more complicated phenomenon. "Weak localization" is a term for a perturbation approach which exists only because we, unlike Nature, are unable to exactly solve the problem of waves in a disordered system in high dimensions.

For both coherent backscatter and Anderson localization, quantitative theory is very difficult. However, coherent backscatter may be understood qualitatively with a simple picture, as shown in Fig. 6. Here a plane wave is incident from a uniform medium onto the surface of a disordered system at some angle from the normal. A ray picture is adopted, and two parts of the plane wave are represented by two rays, q_{in1} and q_{in2}. Because of the obliqueness of the incidence, the two rays arrive at the surface of the disordered medium with a phase difference $\Delta\varphi$. The consequence of the scattering inside the disordered medium may be found by adding all possible internal paths for the rays. The important feature to note is that every path for one ray may be associated with a time reversed path for the other ray, as illustrated in one instance in Fig. 6. Thus outgoing rays q_{out1} and q_{out2} will have undergone the same phase change inside the disordered medium. If the outgoing rays exit at an angle other than the reverse of the incident direction, then the original phase shift $\Delta\varphi$ will not be undone, and if results are averaged, the random phase shift will result in incoherent interference of waves. However, if the outgoing rays are in the backward direction, then the original phase shift $\Delta\varphi$ will also be reversed and all waves will interfere constructively. Thus the averaged intensity in the backward direction will be enhanced relative to other scattering directions.

One very interesting feature of coherent backscatter is

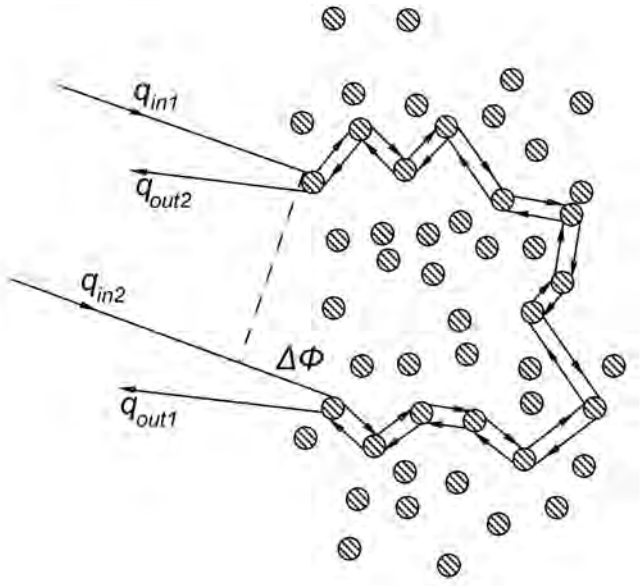

Fig. 6. Qualitative picture of coherent backscatter. Time-reversal invariance results in each internal path having a mate with exactly the opposite phase shift. If the outgoing rays are in the opposite direction from the incident rays, then the phase shift $\Delta\Phi$ is also reversed, and the backscattered rays always add constructively.

that the "incident" wave does not have to originate outside of the disordered system; Weaver[25] has shown that the effect also occurs inside irregular reverberant rooms.

Experimental studies of Anderson localization

Before describing experiments which measure Anderson localization, it would be worthwhile to review some concepts which are important in the design of an experiment. The basic model involves coupled local oscillators—for the periodic system, the local oscillators and the coupling are all identical, and for a disordered system they are different. It is important that the local oscillators and the coupling mechanism have low damping (i.e., have a high quality factor, Q); this is necessary for the waves to have long-range phase coherence, so that interference effects are not degraded.

An important concept for the periodic system is that when beginning with uncoupled local oscillators that have discrete natural frequencies, and when the oscillators are coupled, the discrete frequencies broaden out into bands. The width of the bands (from the lowest frequency to the highest frequency in a band) scales with the strength of the coupling; weak coupling gives rise to narrow bands and wide gaps. There are modes at gap frequencies, but they are evanescent waves which decay exponentially. Suppose disorder is introduced into a periodic system. Now eigenfrequencies may appear in the regions which were formally gaps. Since the modes in the gaps already had exponential decay, expect that modes appearing in the gaps would have eigenfunctions which evolved from the evanescent waves and thus would be strongly localized. A more rigorous treatment by Thouless[26] has shown that this is indeed the case. The conclusion is that it is easier to observe Anderson localization by beginning with narrow bands and wide gaps. Thus for an experiment on Anderson localization, start with weakly coupled local oscillators.

Another crucial undertaking when building an experiment to measure Anderson localization is to first build and test, using the same types of local oscillators and coupling, a periodic system. In this case Bloch's theorem accurately predicts what must be observed, and if the experimental measurements do not match the prediction, then the experiment must be fixed. Only when Bloch's theorem is observed is it possible to confidently proceed to studies of Anderson localization by adding disorder to the system.

In adding disorder to a periodic system, there are two possibilities—put disorder in the local oscillators, or put disorder in the coupling. The former method introduces what is called "diagonal disorder," and the second is called "off-diagonal disorder." From the "balls-and springs" description of how extended eigenfunctions occur in a periodic system, it is expected that diagonal disorder would give stronger Anderson localization, and this is indeed the case. The final conclusion is that to experimentally observe Anderson localization, begin with high Q local oscillators with slightly different natural frequencies, and couple them together weakly.

Acoustic Anderson localization in one dimension

An acoustic experiment which was used for studying waves in a periodic system, as well as a disordered system, was discussed in Part 1 of this series. The experimental system is just like the theoretical model system: a string (steel wire) stretched to some tension, with masses (lead shot) spaced along the string at intervals. Transverse waves are excited in the wire with an electromechanical shaker at one end of the wire. The wave field throughout the system is measured with an electromagnetic pickup which can travel the length of the steel wire, measuring the wave amplitude and phase as a function of position.

The actual system discussed in Part 1 was for lecture demonstrations—the one used for serious research was very similar but much longer, having about 50 masses on a wire about 15 m in length. If the masses were infinite in size, then the local oscillators (the sections of string between the masses) would be uncoupled; thus weak coupling corresponds to large masses. A measure of the size of the masses may be obtained by determining the reflection coefficient at a single mass; the experimentally measured coefficient of 0.9997 indicates that the masses very nearly, but not completely, isolate the local oscillators.

At first the lead shot on the wire were arranged periodically. Since the local oscillator is formed by the length of wire between the masses, calipers were used to insure that the wire lengths between the inside edges of the lead shot were as identical as possible. It should be noted that the size and mass of the individual lead shot were different, with variations of as much as 13%. However, this corresponds to off-diagonal disorder, which does not significantly disturb the otherwise periodic system of local oscillators.

The first step in the experiment was to measure the eigenfrequencies, at least in one band. This was done by placing the pickup near the end of the wire away from the driver, then sweeping the frequency of the drive and monitoring the response of the system with the pickup. Sweeping the fre-

quency corresponds to moving a horizontal line vertically upward in Fig. 2. If the starting frequency is in the gap region, where the modes are exponentially-decaying evanescent waves, then very little response is measured at the far end of the wire. As the frequency (or horizontal line) enters a band, then a sequence of resonances in the pickup is observed as eigenfrequencies (the symbols in the bands in Fig. 2) are traversed. After a whole band is traversed, the response of the system drops while the frequency is in the next gap. Just such features were observed experimentally, as shown in Fig. 7a. Between gap regions of very low response, there are about 50 resonances corresponding to the eigenfrequencies of the 50 mass system. The more slowly varying structure in Fig. 7a is due the pickup being at one point, which is sometimes near a node in the Bloch standing wave.

After measuring the eigenfrequency spectrum, the drive frequency was fixed at one of the resonances (eigenfrequencies) in the band, and the pickup was moved along the length of the wire to measure the eigenfunction. Two eigenfunctions (plotted as wave amplitude as a function of position along the wire) for two different eigenfrequencies are shown in Fig. 8a and 8b. The band which was measured is the second band, for which the eigenfunctions correspond to fitting approximately one half wavelength between the masses; thus in Fig. 8a and 8b, the places where the amplitude drops to nearly zero are the locations of the masses, and the peaks in-between indicate the half wavelengths. The more slowly varying structure in the eigenfunctions is due to the form of the Bloch standing waves. It is important to note that the large amplitude of these eigenfunctions is maintained from the driven end to the far end of the 50 mass system. This extended nature of waves in a periodic system is remarkable given that the reflection coefficient of each mass is 0.9997. The measured results for this system can be compared with the predictions of Floquet's theorem, and it is found that the system of masses and steel wire is an accurate realization of a periodic wave mechanical system.

After measuring the periodic system, the positions of the masses were moved so as to give the system 2% disorder; that is, the lengths of wire between the masses were set to $a(1 + 0.02r)$, where r was a random number between -1 and + 1. The measurement procedure used for the periodic system was then repeated for the disordered system. The measured eigenfrequency spectrum is shown in Fig. 7b. Instead of having a regular pattern, the eigenfrequencies for the disordered system clump into an irregular pattern, and some eigenfrequencies appear at positions which would be in the gap for the periodic system. By setting the drive frequency at one of the new resonances, an eigenfunction for the disordered system could be measured. The eigenfunctions for four different eigenfrequencies are shown in Fig. 8c through 8f. Fig. 8c shows a strongly localized state, and the others are localized to a somewhat lesser degree; this is as predicted because the most localized eigenfunction is the one whose eigenfrequency is well within the gap region (farthest to the left in Fig. 7b).

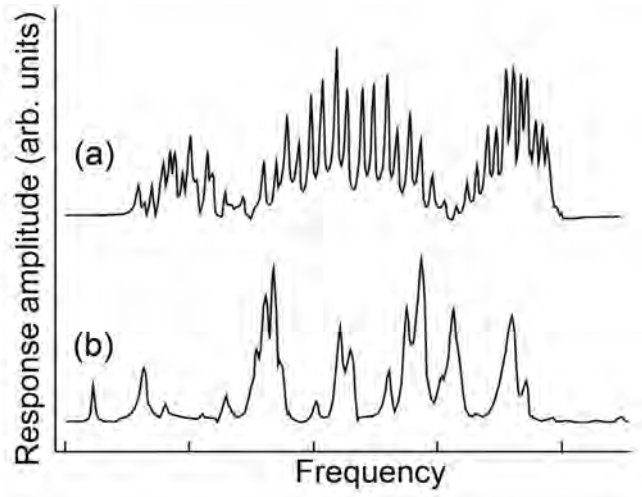

Fig. 7. Measured frequency spectra showing resonances at eigenfrequencies. (a) Spectrum for the steel wire with masses spaced periodically. (b) Spectrum with the mass spacing disordered by 2%; note the eigenfrequency appearing in the gap on the left.

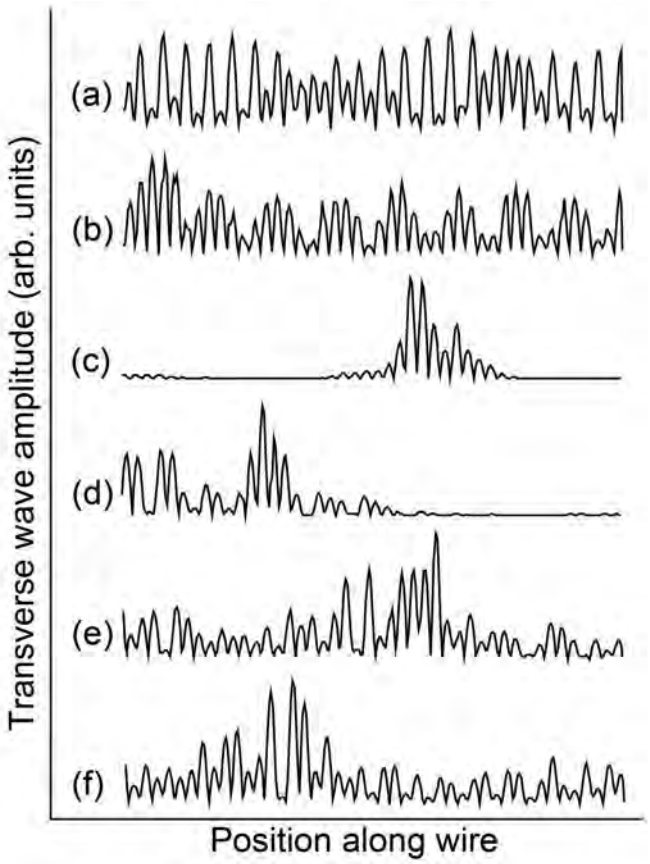

Fig. 8. Eigenfunctions (transverse wave amplitude as a function of position, $\psi(x)$) for different eigenfrequencies. (a) and (b) Eigenfunctions for the periodic system, verifying extended Bloch waves. (c) through (f) Eigenfunctions for the disordered system, showing Anderson localization.

The experiment described above was undertaken to study the effects of a time-dependent potential field on Anderson localization.[27] It turned out that this acoustics experiment provided the first direct experimental observation of an Anderson localized eigenfunction (Fig. 7.)

Classical Anderson localization in higher dimensions

Guided by discussions in the preceding section, a scheme for observing classical Anderson localization in two

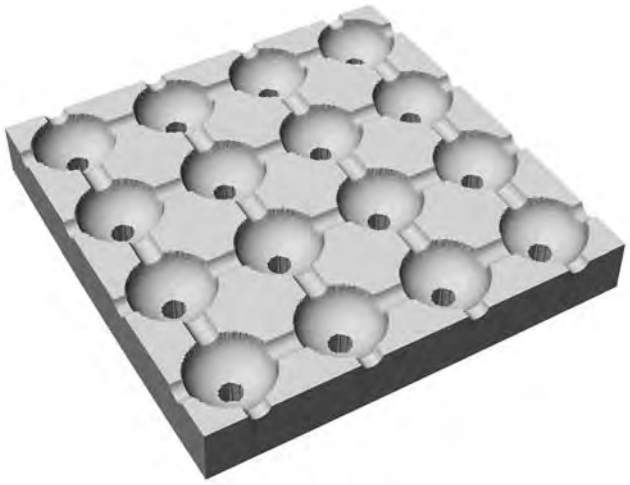

Fig. 9. Illustration of a method for observing Anderson localization in two or three dimensions. Stacking plates machined in this pattern results in an array of coupled Helmholtz oscillators.

or three dimensions would be to begin with an array of high Q local oscillators with slightly different natural frequencies, and then weakly couple them together in a two- or three-dimensional pattern. With weak coupling, a periodic system would have narrow bands and mostly gaps; the introduction of sufficient disorder in the local oscillators would lead to Anderson localized eigenfunctions.

A method for accomplishing this with acoustical waves is illustrated in Fig. 9. A number of aluminum plates (one of which is shown in Fig. 9) are machined on one side with hemispherical depressions, together with interconnecting semi-cylindrical channels. Each plate is machined the same way on the opposite side, and holes are drilled through, connecting the bottoms of the hemispheres. The plates are then stacked and sealed together, resulting in a three dimensional array of spherical cavities connected with cylindrical necks. Acoustically this would be an array of coupled Helmholtz resonators; narrow necks would correspond to weak coupling. The system as described would be periodic, having identical Helmholtz resonators, and could be tested for Bloch wave behavior. For a disordered system, "stuffing blocks" of various sizes could be placed inside the spherical cavities, shifting the local Helmholtz resonance frequencies and introducing diagonal disorder.

There are large numbers of possible systems of two- or three-dimensional arrays of coupled mechanical or electromagnetic oscillators that could be constructed. However, there is little point in actually using these classical systems for serious experimental research. The problem is that following the guidelines to readily observe Anderson localization leads to systems consisting of lumped elements connected with simple coupling, which can be treated exactly in linear theory, and numerical computer calculations of eigenvalues and eigenfunctions could be carried out to high accuracy, much better than could be obtained experimentally. The fundamental physics governing the wave nature of the classical experiments is well established, so that if any significant deviations from the calculated results were observed in an experiment, the only valid conclusion would be that the experiment must be cleaned up. Properly designed experiments could be used for pedagogical purposes, student lab experiments or lecture demonstrations. Classical experiments could also be used to study systems where the theoretical calculations are not so tractable, as for media not consisting of local oscillators, for time-dependent potential fields or for the nonlinear wave equation, as discussed in Part 1[1] of this tutorial series.

In the literature the statement that it is difficult to observe classical Anderson localization is often encountered. In light of the discussion above it would seem that this statement is quite untrue, and in most cases it is incorrect. However, if this statement is made in regard to localization experiments which are motivated by the prospect of practical applications, such as optical waveguide multiplexers, etc., where constraints prevent the use of weakly coupled high Q local oscillators, then obtaining localization effects may be more of an experimental challenge. On the other hand, it is possible to follow the guidelines and be more clever in the design of practical devices utilizing the phenomena of disordered systems, as well as periodic systems. AT

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